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Pushing the Limits: Differential Calculus and Preference Measurement with Large Risks

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Abstract: Differential calculus is a convenient instrument for approximating the rate of change in a function when its argument changes by an infinitesimal increment. Derivatives become inaccurate estimates of change, however, when the increments are large. Yet, a sizeable empirical literature in mainstream economics has begun applying calculus-based metrics to the investigation of attitudes toward large-scale risks. Indeed, a number of recent studies have used game show data for the explicit purpose of measuring absolute and relative risk aversion in the context of major risks. To illustrate the inconsistency, we develop and calibrate models of the computational errors resulting from this misapplication of differential calculus. The results demonstrate why conventional measures of risk aversion and other risk attitudes based on even higher-order derivatives are inappropriate for use with large-scale risks. Given that they are only suitable for use with trivial risks, the conventional metrics would appear to be of little pragmatic value.

Keywords: risk aversion; derivatives; prudence; temperance; edginess

JEL classifications: D81, C00, A12

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1. Introduction

> [T]he peculiarly mathematical conception of infinitesimal quantities is inapplicable to economic problems. The utility afforded by a given amount of commodities, or by a given increase in a given amount of commodities, is either great enough for valuation, or so small that it remains imperceptible to the valuer and cannot therefore affect his judgement. (Ludwig von Mises, 1953, p. 44)

Despite cautionary remarks by von Mises (1953), Rothbard (1963) and others, many economists have been unrelenting in their adoption of convenient but inappropriate mathematical approaches to analyzing economic problems. Without attempting to address this broad issue generally, the present paper critically examines the use of mathematics in a particular economic application. Specifically, we illustrate the mismeasurement of preferences over large risks that can be engendered by employing differential calculus in utility analysis.

For this purpose, we set aside deeper philosophical issues such as whether utility functions can be cardinal in nature (on this, see Barnett, 2003). By focusing strictly on computational matters, we demonstrate that, even if one accepts the theoretical assumptions of conventional models, they are eminently unsuitable for use with risks of meaningful magnitude. This, then, begs the question of whether models designed exclusively for measuring attitudes toward trivial risks have any economic importance.

The following section provides background information on the conventional risk aversion metrics developed through the use of differential calculus, and the recent empirical applications of those metrics to large-scale risks. We then present our primary analysis. Using the most commonly applied utility functions, we estimate the magnitude of potential computational errors in calculus-based risk aversion estimates when the limits imposed by differential calculus are ignored and large wealth changes, as opposed to minute increments, are involved. The analysis is then extended to risk preference measures involving higher-order derivatives, and the paper ends with a short conclusion.

2. Background

Aversion to risk has been an important topic in economics since Daniel Bernoulli (1738/1954) invoked it to explain the St. Petersburg Paradox, but it has only been quantified since the 1960s, when Arrow (1964) and Pratt (1965) first introduced their measures of absolute and relative risk aversion. As they did so, both Arrow (1964) and Pratt (1965) were careful to point out the limitations of their analysis. In particular, each recognized that their employment of differential calculus restricted their models to infinitesimal changes in wealth. Arrow (1964, p. 34) observed, "The absolute risk aversion directly measures the insistence of an individual for more-than-fair odds, *at least when the bets are small*" [emphasis added]. Pratt (1965, p. 126) interpreted the same coefficient as "a measure of the *local risk aversion* or *local propensity to insure* at the point x under the utility function" [emphasis in original], equal to "twice the actuarial value the decision maker requires per unit of variance for infinitesimal risks" and added, "we have not introduced any measure of risk aversion in the large."

The insistence of both Arrow (1964) and Pratt (1965) that their measures applied exclusively to miniscule risks can be attributed to their use of the first and second derivatives of utility. Indeed, an earlier generation referred to the mathematics of limits and derivatives not as differential calculus, but as infinitesimal calculus.[1] For a twice-differentiable utility function $u(x)$ defined over wealth of x, the Arrow-Pratt coefficient of relative risk aversion is the point-elasticity of marginal utility with respect to wealth, or

$$r(x) = -xu''(x)/u'(x) \tag{1}$$

where $u'(x)$ and $u''(x)$ denote the first and second derivatives of utility, respectively. The coefficient of absolute risk aversion is similar, but unweighted by initial wealth:

$$a(x) = -u''(x)/u'(x)\,. \tag{2}$$

Whereas $a(x)$ is intended to measure an individual's reluctance to place a fixed amount of wealth at risk, $r(x)$ is meant to measure his or her reluctance to have a given proportion of wealth subjected to risk (Arrow, 1964; Pratt, 1965).

Over the ensuing five decades, scores of empirical studies have estimated the Arrow-Pratt coefficients in a wide variety of contexts, including consumption, investments, insurance, labor supply, entrepreneurship, and others, with a variety of estimation techniques and a wide range of reported outcomes. A number of survey papers, including those by Meyer and Meyer (2006), Cox and Harrison (2008), and Outreville (2014), have already reviewed the empirical literature. Unfortunately, the subsequent research has often been less cautious than Arrow (1964) and Pratt (1965) when measuring risk aversion, neglecting the limitations imposed by the calculus, and instead applying the Arrow-Pratt metrics to risks of sizeable magnitude. Indeed, a growing number of recent studies have intentionally selected data sets involving large-scale gambles for the purpose of estimating the Arrow-Pratt coefficients of absolute and relative risk aversion. In particular, televised game shows have frequently been used as data sources, because such shows provide the opportunity to observe the behavior of individual decision-makers when faced with sizeable wagers, both in absolute terms and relative to most contestants' income or initial wealth. As Gertner (1993) and others have noted, the potential payouts offered by game shows are typically orders of magnitude larger than the budgets of experimental researchers who attempt to simulate lotteries in laboratories. Additionally, the parameters of the game show risks—such as the probability of winning and the magnitude of the payoff—can readily be identified, data collection is relatively simple (often being provided by the producers of the shows), the wealth changes experienced by contestants are real rather than hypothetical, and each show is unique, making each study somewhat novel. Several authors have even referred to these shows as natural experiments for studying risk aversion (Metrick, 1995; Beetsma and Schotman, 2001). The appendix briefly reviews the risk aversion estimates in eleven such studies from around the world, along with the payouts on the game shows from which the estimates were derived.

Despite differences in game show rules, populations, currency units, estimation techniques, and results, these studies can be summarized as follows. First, they involve gambles that are acknowledged by their authors to constitute large-scale wagers, with potential payoffs that are often at least as large as the estimated level of a player's annual income, and up to 20 or even 50 times the assumed level of a player's total initial wealth. Second, they typically assume that contestants exhibit either exponential or isoelastic utility functions.[2] Third, the estimated values of $a(x)$ range over several orders of magnitude, from approximately $6x10^{-7}$ to $2.0x10^{-3}$, while the estimates of $r(x)$ range from approximately 0.30 to 13, with most $r(x)$ values being in the single digits. Given the nature of televised game shows, however, one might expect that the contestants, who voluntarily choose to participate in these games of chance, are inherently less risk averse than the general population.

Of course, these are not the only studies that have employed Arrow-Pratt metrics in the context of large wealth fluctuations.[3] The game show literature, however, provides a convenient set of calibrations with which we will illustrate why such enormous gambles are an inappropriate setting for use of the Arrow-Pratt coefficients.

Our objective is neither to critique nor to replicate the various methodologies used for estimation in those prior studies of risk aversion. Rather, our intent is to illustrate more generally why data sets involving large risks are unsuited to calculus-based metrics such as the Arrow-Pratt coefficients, or equivalently, why calculus-based metrics are inapplicable to large risks.

3. Absolute and Relative Risk Aversion

To appreciate the following analysis, it is important to keep in mind how the derivatives of utility are constructed. By definition, the derivative is the rate of change in a function as its argument changes by an infinitesimal amount. That is, it measures the slope of the function when the increment (denoted here by ε) approaches zero in the limit (Rhode, *et al*., 2012). Thus, marginal utility, or the first derivative of utility with respect to wealth, is [4]

$$u'(x) = \frac{du}{dx} = \lim_{\varepsilon \to 0} \frac{u(x+\varepsilon) - u(x)}{\varepsilon}. \tag{3}$$

Assuming that the utility function is at least twice differentiable, the second derivative shows how $u'(x)$ changes when x is increased by the increment ε:

$$u''(x) = \frac{d^2u}{dx^2} = \lim_{\varepsilon \to 0} \frac{u'(x+\varepsilon) - u'(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{[u(x+2\varepsilon) - u(x+\varepsilon)] - [u(x+\varepsilon) - u(x)]}{\varepsilon^2}$$

$$= \lim_{\varepsilon \to 0} \frac{u(x+2\varepsilon) - 2u(x+\varepsilon) + u(x)}{\varepsilon^2}. \tag{4}$$

In both equations (3) and (4), "the limit as ε goes to zero" is an important restriction, and it explains why Arrow (1964) and Pratt (1965) carefully described their models as applicable only when the risks are infinitesimal. We may think of the derivatives of utility as convenient shortcuts that simplify the computation of a rate of change when ε is small, but not when ε is large; as ε increases, these shortcuts become less accurate. As a simple example, consider the (risk-loving) utility function $u(x) = x^3$, for which $u'(x) = \lim_{\varepsilon \to 0} 3x^2 + 3x\varepsilon + \varepsilon^2$. In the limit, with $\varepsilon \approx 0$, the local approximation of marginal utility reduces to $3x^2$; but in the absence of calculus, we would compute marginal utility for an increment of ε as $3x^2 + 3x\varepsilon + \varepsilon^2$; as ε increases, the difference between these measures, $3x\varepsilon + \varepsilon^2$, becomes increasingly significant. Likewise, $u''(x) = 6x$ in the limit as ε approaches zero, but a direct measurement of the second-order utility change gives $6x + 6\varepsilon$; the two values clearly diverge as ε deviates from zero. For a visual illustration in the context of a utility function displaying risk aversion, consider Figure 1, where the slope of the utility curve at a single point A (as when the risk is small) clearly differs from the slope between two distant points A and B (as when the risk is large).

To see the effect that such differences can have on estimates of risk aversion, consider measures that are analogous to equations (1) and (2) but that ignore the limits in equations (3) and (4) and thus allow ε to take nontrivial values. That is, suppose we were to measure absolute risk aversion (by analogy to the Arrow-Pratt metrics) as the (negative) rate of change in marginal utility, divided by marginal utility for a particular increment of wealth, where rates of change are not measured by derivatives. Dividing the right-hand side of equation (4) by the right-hand side of equation (3), but disregarding the limits and multiplying the resulting ratio by –1, gives what might be called an "unlimited" analogue of $a(x)$ as

$$A(x,\varepsilon)=\frac{-1}{\varepsilon}\left[\frac{u(x+2\varepsilon)-2u(x+\varepsilon)+u(x)}{u(x+\varepsilon)-u(x)}\right]=\frac{1}{\varepsilon}\left[1+\frac{u(x+\varepsilon)-u(x+2\varepsilon)}{u(x+\varepsilon)-u(x)}\right]. \tag{5}$$

Multiplying equation (5) by x gives an unlimited analogue of $r(x)$ as

$$R(x,\varepsilon)=\frac{-x}{\varepsilon}\left[\frac{u(x+2\varepsilon)-2u(x+\varepsilon)+u(x)}{u(x+\varepsilon)-u(x)}\right]=\frac{x}{\varepsilon}\left[1+\frac{u(x+\varepsilon)-u(x+2\varepsilon)}{u(x+\varepsilon)-u(x)}\right]. \tag{6}$$

We may think of $A(x, \varepsilon)$ and $R(x, \varepsilon)$ as the target values approximated by $a(x)$ and $r(x)$, respectively.

3.1 Absolute Risk Aversion

We now examine how $a(x)$ differs from $A(x, \varepsilon)$ under various parameterizations of exponential utility as found in the literature, and with varying values of x and ε. Suppose an empirical investigation reveals (or assumes) that the behavior of an individual facing a risky change in wealth of 100 units is best described by the negative exponential function, $u(x)=-EXP(-\gamma x)$. This is often called the constant absolute risk aversion (CARA) function, because the first and second derivatives yield $a(x) = \gamma$ regardless of x.[5] Suppose further that γ is estimated to be 0.0003, perhaps on the basis of the individual's certainty equivalent for the risk at hand. Finally, imagine that the behavior of a second individual, facing a risky change in wealth of 5,000 units, is best described by the same utility function, also with $\gamma = 0.0003$. Then the Arrow-Pratt coefficient of absolute risk aversion is calculated to be $a(x) = 0.0003$ in both cases, but a direct measurement of utility changes using $A(x, \varepsilon)$ will reveal a difference between the two individuals, as shown below.

Substituting the negative exponential utility function into equation (5) gives

$$A(x,\varepsilon)=\frac{-1}{\varepsilon}\left[\frac{-e^{-\gamma(x+2\varepsilon)}+2e^{-\gamma(x+\varepsilon)}-e^{-\gamma x}}{-e^{-\gamma(x+\varepsilon)}+e^{-\gamma x}}\right]=\frac{-1}{\varepsilon}\left[\frac{e^{-2\gamma\varepsilon}-2e^{-\gamma\varepsilon}+1}{e^{-\gamma\varepsilon}-1}\right]. \tag{7}$$

It is clear from equation (7) that initial wealth, x, drops out of the calculation, so the result depends only on the increment to wealth (ε) and the curvature of the utility function, as captured by γ. For the individual whose ε is only 100 units, equation (7) yields $A(x, \varepsilon) = 0.000296$; the difference between $a(x)$ and $A(x, \varepsilon)$ is less than two percent of the latter. In that case, nearly the same result is obtained by taking derivatives of the utility function as by inserting initial and incremental wealth values directly into the utility function to compute marginal utility and its rate of change. But for the individual who faces the larger (ε = 5,000) change in wealth, $A(x, \varepsilon)$ = 0.000155, only about half as large as the calculus-based estimate $a(x)$. With this larger wealth increment, derivatives no longer provide reasonable shortcuts for computation because "the limit as ε goes to zero" does not apply, and the Arrow-Pratt coefficient loses accuracy as a measure of the individual's preferences. The difference between $a(x)$ and $A(x, \varepsilon)$ constitutes a computational error resulting from the application of differential calculus rather than direct measurement of utility changes.

Table 1 gives additional comparisons between $a(x)$ and $A(x, \varepsilon)$ for various values of ε and x, and selected values of γ suggested by the empirical studies reviewed in the appendix. The table confirms that while initial wealth is irrelevant when utility is exponential, the computational errors increase with the curvature of utility and the increment to wealth. That is, as the individual's preferences deviate from risk neutrality and/or the risk becomes large, the computational errors arising from calculus increase. Note that in each case, $a(x)$ overestimates $A(x, \varepsilon)$. The magnitude of the error approaches five percent of $A(x, \varepsilon)$ when the multiplicative product of γ and ε exceeds roughly 0.1, as when γ = 0.000005 and ε = 20,000; errors exceeding five percent are shown in bold font. This suggests that "large risks" cannot be defined independently of risk aversion; a given wager may be small enough for the Arrow-Pratt metrics if the individual is nearly risk neutral, but too large for the Arrow-Pratt measures to produce accurate results if the individual is highly risk averse. In an exponential utility function with γ = 0.002, for example, the computational error exceeds ten percent of the target value at a wealth increment of just 100 units. Of course, many of the stakes in televised game shows exceed even the highest values of ε shown in Table 1, and would thus incur even larger computational errors.[6]

3.2 Relative Risk Aversion

We next conduct a similar analysis for isoelastic utility, defined as $u(x) = (x^{1-\beta})/(1-\beta)$ for all $\beta \neq 1$, and $u(x) = \ln x$ otherwise. Because the first and second derivatives yield $r(x) = \beta$ regardless of x, this is commonly referred to as the constant relative risk aversion (CRRA) function.

As a concrete example, suppose an individual faces a 50-50 gamble to win or lose ε, so expected utility is $u(x-\pi) = 0.5u(x+\varepsilon) + 0.5u(x-\varepsilon)$, where π denotes the risk premium. Now suppose initial wealth is x = 20,000, the bet is ε = 10,000 and the risk premium is π = 2,500. Then it is easily verified that a logarithmic function fits the expected utility model well: $\ln 17{,}500 \approx 0.5 \ln 30{,}000 + 0.5 \ln 10{,}000$, and an observer applying calculus to the utility function would obtain $r(x)$ = 1. But inserting logarithmic utility into equation (6) yields $R(x, \varepsilon)$ = 0.58; the Arrow-Pratt metric $r(x)$ overestimates the target by 72 percent. Table 2 provides additional illustrations using selected values of β suggested by the empirical studies reviewed above, and the same initial and incremental wealth values used in Table 1.

The major difference between the CARA function and the CRRA function is that, in the latter, it is the proportion of wealth at risk, rather than the absolute amount of wealth at risk, that matters.

For any given specification of isoelastic utility, the computational error depends only on the ratio of ε to x. Notice, for example, that for $\beta = 0.5$, the error exceeds 59 percent of $R(x, \varepsilon)$ whenever $\varepsilon/x = 1/2$, and (as with CARA) the computational errors increase with the curvature of utility, so the greater the degree of risk aversion, the less accurate the calculus-based estimates are. With $\beta = 2.5$, for example, $\varepsilon/x = 1/2$ yields an error of more than 115 percent. These would appear to be unacceptably high levels of inaccuracy for empirical work, and they illustrate why the Arrow-Pratt coefficients should be restricted exclusively to minor risks.

Indeed, for $\beta = 0.5$, an error exceeding five percent of the target value occurs when ε is 1/20 of x, and with $\beta = 7$, a five percent error occurs when ε is only 1/80 of x. Thus, if "small risks" are interpreted as those for which calculus is appropriate, generating computational errors of less than five percent, then for isoelastic utility, the threshold is probably no higher than 1/20 of initial wealth, and closer to 1/80 or less of initial wealth for more risk averse individuals.[7]

4. Higher-Order Risk Preferences

Although risk aversion is by far the most empirically studied risk preference, several additional attitudinal metrics that rely on differential calculus have been developed over the past quarter-century. Kimball (1990) introduced the term *prudence* to describe the attitude leading to precautionary saving, and defined relative prudence as $p(x) = -xu'''(x)/u''(x)$. *Temperance* refers to the tendency to moderate overall portfolio risk by reducing risk exposure in one area when confronted by an unavoidable risk elsewhere (Kimball, 1992; Eeckhoudt, *et al*., 1996); relative temperance is measured by $t(x) = -xu''''(x)/u'''(x)$. And Lajeri-Chaherli (2004) coined the term *edginess* to describe the attitude of an individual who remains prudent in the presence of an independent background risk; relative edginess is measured by $g(x) = -xu'''''(x)/u''''(x)$. Each of these metrics attempts to describe the responsiveness of an individual decision-maker to an exogenous source of detrimental risk (Denuit and Rey, 2010). In each case, $-u^{(n)}(x)/u^{(n-1)}(x)$ is the absolute measure, and $-xu^{(n)}(x)/u^{(n-1)}(x)$ is the relative measure, where $u^{(n)}(x)$ denotes the nth derivative of utility with respect to x. Of course, these models assume not only that the changes to wealth are miniscule, but also that utility is differentiable to at least the nth order.[8]

Compared with the enormous literature on risk aversion, relatively few studies have undertaken empirical estimation of these metrics. Using consumption data, Dynan (1993) estimated values of p(x) from 0.024 to 0.292, though a replication taking account of liquidity constraints obtained values closer to, and in some cases exceeding, one (Lee and Sawada, 2007). With life insurance data, Eisenhauer (2000) found values of $p(x)$ ranging between 1.5 and 5.15. In a laboratory experiment, Bostian and Heinzel (2011) reported a mean value of $r(x)$ equal to 2.06 and a mean value of $p(x)$ equal to 3.90. For a lottery experiment with individuals having wealth of approximately €23,325 and facing payouts of up to €12,960, Noussair, *et al*. (2014) estimated relative risk aversion to be near 3.38, relative prudence near 3.45, and relative temperance near 3.00.[9] To date, it appears that no empirical estimates of edginess or higher-order risk apportionments have been undertaken.

We now examine what discrepancies may occur if metrics such as $p(x)$, $t(x)$, and $g(x)$ are applied to risks of large magnitude. We therefore wish to find, as we did for risk aversion, unlimited analogues of prudence, temperance, and edginess, and calibrate them with large values of ε.

Generalizing equations (3) and (4) to higher order derivatives, we can write the *n*th derivative of utility as

$$u^{(n)}(x) = \frac{d^n u}{dx^n} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \sum_{k=0}^{n} \left[\frac{(-1)^{n-k} n!}{k!(n-k)!} \right] u(x + k\varepsilon) . \tag{8}$$

Let us again ignore the limit in each case, and instead consider changes in utility for ε of any size. Then the analogue of $p(x)$, relative prudence, is given by

$$P(x,\varepsilon) = \frac{-x}{\varepsilon} \left[\frac{u(x+3\varepsilon) - 3u(x+2\varepsilon) + 3u(x+\varepsilon) - u(x)}{u(x+2\varepsilon) - 2u(x+\varepsilon) + u(x)} \right]. \tag{9}$$

Similarly, an analogue of relative temperance $t(x)$ can be constructed as

$$T(x,\varepsilon) = \frac{-x}{\varepsilon} \left[\frac{u(x+4\varepsilon) - 4u(x+3\varepsilon) + 6u(x+2\varepsilon) - 4u(x+\varepsilon) + u(x)}{u(x+3\varepsilon) - 3u(x+2\varepsilon) + 3u(x+\varepsilon) - u(x)} \right], \tag{10}$$

and the analogue of relative edginess, $g(x)$, is

$$G(x,\varepsilon) = \frac{-x}{\varepsilon} \left[\frac{u(x+5\varepsilon) - 5u(x+4\varepsilon) + 10u(x+3\varepsilon) - 10u(x+2\varepsilon) + 5u(x+\varepsilon) - u(x)}{u(x+4\varepsilon) - 4u(x+3\varepsilon) + 6u(x+2\varepsilon) - 4u(x+\varepsilon) + u(x)} \right]. \tag{11}$$

Proceeding as we did in the case of risk aversion, we calibrate the differences between $p(x)$ and $P(x, \varepsilon)$, between $t(x)$ and $T(x, \varepsilon)$, and between $g(x)$ and $G(x, \varepsilon)$, for the levels of x and ε used in Tables 1 and 2. With any isoelastic utility function, $p(x) = r(x) + 1$, $t(x) = r(x) + 2$, and $g(x) = r(x) + 3$. For simplicity, Table 3 illustrates the computational errors using logarithmic utility (that is, CRRA with $\beta = 1$), for which $r(x) = 1$, $p(x) = 2$, $t(x) = 3$, and $g(x) = 4$. The third and fourth columns of Table 3 replicate the fifth and sixth columns, respectively, of Table 2, and the remaining columns of Table 3 illustrate the dimensions of computational errors for higher-order preferences. Notice that, for any given combination of x and ε, the errors resulting from the use of calculus increase with the order of the derivatives involved. For example, when ε is half as large as x, the computational error associated with $r(x)$ is approximately 72 percent, the error from $p(x)$ is about 121 percent, the error from $t(x)$ is 170 percent, and the error involved in using $g(x)$ would be nearly 220 percent of the respective target values. Thus, if a risk is too large for use of the Arrow-Pratt metrics, it is also too large for related measures such as prudence, temperance, or edginess.[10]

5. Conclusion

The study of attitudes toward large-scale risks is undeniably important, and there is nothing inherently wrong with using game show data, or other large gambles, to investigate risk preferences. The problem lies in applying the Arrow-Pratt coefficients, or other metrics that are based on derivatives, to measure risk attitudes in such contexts.

It has been said that calculus is "probably the most effective instrument for scientific investigation that mathematics has ever produced" (Boyer, 1949, p. 186). But even if that is true, as a scientific instrument, differential calculus is essentially analogous to a microscope, designed for studying the effects of minute changes in the argument of a function; it is assuredly not suitable for use with large changes. Even undergraduate textbooks frequently distinguish between the (calculus-based) point elasticity and the (algebraic) arc elasticity in contexts such as the price elasticity of demand; yet the distinction is widely neglected in the measurement of risk preferences, where the point elasticity of marginal utility with respect to wealth is commonly used regardless of the magnitudes involved. Our analysis suggests that calculus is a convenient computational shortcut for measuring marginal utility and its rate of change only if wealth increments are small and preferences are close to risk neutral; computational errors increase with the size of the wealth increment, the curvature of utility, and the order of derivatives being used to estimate preferences. Our analysis also suggests that "large" risks cannot be defined separately from preferences; if large risks are taken to be those for which the computational errors from calculus are greater than, say, five percent, then the threshold at which stakes are deemed to be large declines as risk aversion increases.

Of course, even more fundamental criticisms of the conventional metrics can be raised. The Arrow-Pratt coefficients and their related measures beg the question of whether utility can even be described by a mathematical function, and if so, whether such a function is cardinal and sufficiently differentiable to calculate derivatives of the requisite order. Setting those concerns aside, however, the present note demonstrates that, even if one accepts the neoclassical assumption that utility can be expressed as a highly differentiable function of wealth, employing derivatives in the context of major risks leads to severe inaccuracies.

As von Mises (1953) had pointed out a decade before the Arrow-Pratt metrics appeared in the literature, quantities that are small enough to justify the use of differential calculus may well be negligible or irrelevant to a decision maker. Rothbard (1963, p. 306) echoed that view, writing, "The human being cannot see the infinitely small step, and it therefore has no meaning to him and no relevance to his action." This suggests that calculus-based measures, properly applied only to infinitesimal risks, may be of little or no practical value, regardless of how theoretically interesting they may be. Indeed, seeking to derive and empirically estimate behavioral implications from indices based on the second, third, fourth, fifth, and even higher-order derivatives of utility at a single point on the curve seems rather like asking, as medieval philosophers are purported to have done, how many angels can dance on the head of a pin. Accurate investigation of attitudes toward meaningful, large-scale risks will require alternative metrics that do not rely on differential calculus.

This analysis may also have implications that go beyond the measurement of risk preferences. Whether differential calculus has caused computational errors in the measurement of other large-scale economic phenomena is an interesting question for future research.

Appendix: Risk Aversion Estimates from Game Shows

This appendix reviews the international literature that has undertaken risk preference measurements on the basis of televised game shows. The payouts and empirical risk aversion estimates are used to calibrate the models of computational errors in Tables 1 through 3.

Using the game show *Card Sharks*, where stakes could reach $16,000 at a time when that amount was larger than the per capita personal income in the United States, Gertner (1993) estimated the lower bound for $a(x)$ in an exponential utility function to be between 7×10^{-5} and 3.1×10^{-4}. Metrick (1995) likewise assumed an exponential utility function for players on *Jeopardy*, where payouts could exceed $50,000; he obtained $a(x)$ values from 3.7×10^{-5} to 6.6×10^{-5}. Hersch and McDougall (1997) studied the game show *Illinois Instant Riches*, in which players could win prizes of up to $100,000; within their sample, median household income ranged from $10,000 to $181,800. Assuming exponential utility, they found $a(x)$ values ranging from 2.6×10^{-5} to 5.6×10^{-5}. Fullenkamp, *et al*. (2003) examined the game show *Hoosier Millionaire*, where payouts could reach $1 million. With per capita income of $11,000 to $14,000 at the time, contestants were assumed to have initial wealth of either $20,000 or $48,000; exponential utility yielded estimates of $a(x)$ ranging from 4.8×10^{-6} to 9.7×10^{-6}, and isoelastic utility yielded estimates of $r(x)$ ranging from 0.63 to 1.76.

Similar analyses have been undertaken around the world. Beetsma and Schotman (2001) applied both exponential and isoelastic utility functions to data from the Dutch game show *Lingo*, where prizes could reach 10,000 Dutch guilders. With initial wealth levels ranging from zero to *f*100,000 they estimated $r(x)$ to range from 0.42 to 13.08; and taking median household wealth in the Netherlands as approximately *f*50,000, they found that their best estimate of $r(x)$ was 7. Deck *et al*. (2008) studied game show players in Mexico, where initial wealth levels were assumed to range from 89,000 to 210,000 pesos, and prizes could reach 5 million pesos. Assuming exponential utility, they obtained $a(x)$ values from 6×10^{-7} to 4.8×10^{-6}, and for isoelastic utility, they estimated $r(x)$ values between 0.3 and 2.6. Post *et al*. (2008) studied *Deal or No Deal* in three countries; they assumed an expo-power function, but found that in Germany, it reduced to exponential utility, for which $a(x)$ was about $1.6\times e^{-5}$. In a large-stakes experiment designed to mimic the game, they estimated $a(x)$ to be 0.002. Examining 298 players on the Italian game show *Affari Tuoi*, where payouts could reach €500,000, Botti, *et al*. (2008) estimated $a(x)$ of about 0.01 with exponential utility, and estimated $r(x)$ to be around 0.4 using isoelastic utility. De Roos and Sarafidis (2010) used data from *Deal or No Deal* in Australia, where contestants could win up to 50,000 Australian dollars, which exceeded the average annual income at the time. Assuming exponential utility, they estimated $a(x)$ to range from 2.7×10^{-5} to 5.7×10^{-5}, and using isoelastic utility, they found $r(x)$ values between 1.8 and 3.2. Observing contestants on *Affari Tuoi*, who had mean labor income of €18,000 and were assumed to have initial wealth of €180,000, Bombardini and Trebbi (2012) defined large-stakes gambles as those for which the expected value was at least

€75,000 or alternatively, a single prize could exceed €250,000; isoelastic utility yielded estimates of $r(x)$ from approximately 0.9 in a static model to 1.4 in a dynamic model. Most recently, using data from the show *Who Wants to be a Millionaire*? in the United Kingdom, with prizes of up to £1,000,000, Hartley *et al*. (2014) found utility to be approximately logarithmic, implying $r(x)$ values near 1.[11]

Endnotes

1. Irving Fisher (1921), for example, titled his review of mathematics for economists *A Brief Introduction to the Infinitesimal Calculus*, and as late as World War II, Boyer (1949, p. 157) noted, "Even now the subject is generally known as the 'infinitesimal calculus'."

2. Indeed, in a review of more than two dozen empirical studies of risk aversion from 1976 to 2012, Eisenhauer (2016) found that about fifty percent of the utility functions were isoelastic, approximately one-quarter were exponential, and the remaining quarter consisted of quadratic and other functional forms.

3. Light and Ahu (2004), for example, argued that divorce constitutes a high-stakes gamble, and used an isoelastic utility function with hypothetical lottery questions to estimate the coefficient of relative risk aversion, which was then used as a predictor of marital dissolution.

4. One can calculate a "right-hand" derivative if ε denotes a gain, a "left-hand" derivative for a loss, or a symmetric derivative with an equal gain and loss. Because game show contestants never actually lose their initial wealth, we simplify the analysis by focusing on right-hand derivatives.

5. The very fact that it is commonly called the CARA utility function suggests just how influential the Arrow-Pratt coefficient of absolute risk aversion has become in the literature. Eisenhauer (2016) examines the specification error involved in assuming an incorrect functional form.

6. Moreover, the ratio of a second-order change to a first-order change degenerates for CARA utility with large stakes. Notice that, for sufficiently large stakes, the increment to wealth becomes the dominant factor in the calculation of $A(x, \varepsilon)$: as $\gamma\varepsilon$ becomes large, the negative exponential terms on the far right-hand side of equation (7) become extraordinarily small, so $A(x, \varepsilon)$ becomes approximately equal to $1/\varepsilon$. Ironically, such a value conveys essentially no information about the individual's preferences. Depending upon the number of decimal places to which one is willing to carry the calculation, this result may be said to occur when $\gamma\varepsilon$ exceeds 10; as Table 1 shows, for example, when $\gamma = .002$ and $\varepsilon \geq 5{,}000$, equation (7) yields $A(x, \varepsilon) \approx 1/\varepsilon$.

7. Note further that, as the β parameter in a CRRA function increases and ε/x rises, $R(x, \varepsilon)$ approaches x/ε; for example, if $\beta = 7$, $x = 10{,}000$ and $\varepsilon = 80{,}000$, then $R(x, \varepsilon) \approx 1/8$.

8. Deck and Schlesinger (2014, p. 1919) have gone even further than edginess, proposing that the sixth derivative of utility be used to construct a preference measure called "risk apportionment of order 6".

9. Others, such as Arrondel, *et al*. (2010) and Ebert and Weisen (2011), have found evidence of prudence and temperance, but did not estimate reduced-form measures of $p(x)$ or $t(x)$.

10. Indeed, this set of calibrations is rather conservative. Because the computational errors increase with the curvature of utility, if we were to parameterize isoelastic utility with values of $\beta > 1$, the errors in Table 3 would increase accordingly.

11. Other studies, such as those by Brooks, *et al*. (2009) and Hopland, *et al*. (2013) use game show data to investigate risk aversion, but do not estimate the Arrow-Pratt coefficients.

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Table 1. Computational Errors from Absolute Risk Aversion with Exponential (CARA) Utility[a]

		$\gamma = a(x) = .000005$		$\gamma = a(x) = .00004$		$\gamma = a(x) = .0003$		$\gamma = a(x) = .002$	
x	ε	$A(x, \varepsilon)$	%Error	$A(x, \varepsilon)$	%Error	$A(x, \varepsilon)$	%Error	$A(x, \varepsilon)$	%Error
10,000	100	.000004999	0.03	.0000399	0.20	.0002955	1.51	.0018127	**10.33**
10,000	1,000	.000004988	0.25	.0000392	2.01	.0002592	**15.75**	.0008647	**131.30**
10,000	5,000	.000004938	1.26	.0000363	**10.33**	.0001554	**93.08**	.0002000	**900.05**
10,000	10,000	.000004877	2.52	.0000330	**21.33**	.0000950	**215.72**	.0001000	**1900.00**
10,000	20,000	.000004758	**5.08**	.0000275	**45.28**	.0000499	**501.49**	.0000500	**3900.00**
10,000	40,000	.000004532	**10.33**	.0000200	**100.48**	.0000250	**1100.01**	.0000250	**7900.00**
10,000	80,000	.000004121	**21.33**	.0000120	**233.60**	.0000125	**2300.00**	.0000125	**15900.00**
20,000	100	.000004999	0.03	.0000399	0.20	.0002955	1.51	.0018127	**10.33**
20,000	1,000	.000004988	0.25	.0000392	2.01	.0002592	**15.75**	.0008647	**131.30**
20,000	5,000	.000004938	1.26	.0000363	**10.33**	.0001554	**93.08**	.0002000	**900.05**
20,000	10,000	.000004877	2.52	.0000330	**21.33**	.0000950	**215.72**	.0001000	**1900.00**
20,000	20,000	.000004758	**5.08**	.0000275	**45.28**	.0000499	**501.49**	.0000500	**3900.00**
20,000	40,000	.000004532	**10.33**	.0000200	**100.48**	.0000250	**1100.01**	.0000250	**7900.00**
20,000	80,000	.000004121	**21.33**	.0000120	**233.60**	.0000125	**2300.00**	.0000125	**15900.00**
40,000	100	.000004999	0.03	.0000399	0.20	.0002955	1.51	.0018127	**10.33**
40,000	1,000	.000004988	0.25	.0000392	2.01	.0002592	**15.75**	.0008647	**131.30**
40,000	5,000	.000004938	1.26	.0000363	**10.33**	.0001554	**93.08**	.0002000	**900.05**
40,000	10,000	.000004877	2.52	.0000330	**21.33**	.0000950	**215.72**	.0001000	**1900.00**
40,000	20,000	.000004758	**5.08**	.0000275	**45.28**	.0000499	**501.49**	.0000500	**3900.00**
40,000	40,000	.000004532	**10.33**	.0000200	**100.48**	.0000250	**1100.01**	.0000250	**7900.00**
40,000	80,000	.000004121	**21.33**	.0000120	**233.60**	.0000125	**2300.00**	.0000125	**15900.00**
80,000	100	.000004999	0.03	.0000399	0.20	.0002955	1.51	.0018127	**10.33**
80,000	1,000	.000004988	0.25	.0000392	2.01	.0002592	**15.75**	.0008647	**131.30**

80,000	5,000	.000004938	1.26	.0000363	**10.33**	.0001554	**93.08**	.0002000	**900.05**
80,000	10,000	.000004877	2.52	.0000330	**21.33**	.0000950	**215.72**	.0001000	**1900.00**
80,000	20,000	.000004758	**5.08**	.0000275	**45.28**	.0000499	**501.49**	.0000500	**3900.00**
80,000	40,000	.000004532	**10.33**	.0000200	**100.48**	.0000250	**1100.01**	.0000250	**7900.00**
80,000	80,000	.000004121	**21.33**	.0000120	**233.60**	.0000125	**2300.00**	.0000125	**15900.00**

[a] Errors exceeding five percent of the target value are shown in bold font.

Table 2. Computational Errors from Relative Risk Aversion with Isoelastic (CRRA) Utility[a]

		$\beta = r(x) = 0.5$		$\beta = r(x) = 1$		$\beta = r(x) = 2.5$		$\beta = r(x) = 7$	
x	ε	$R(x, \varepsilon)$	%Error	$R(x, \varepsilon)$	%Error	$R(x, \varepsilon)$	%Error	$R(x, \varepsilon)$	%Error
10,000	100	.49384	1.25	.98524	1.50	2.44495	2.25	6.69645	4.53
10,000	1,000	.44512	**12.33**	.87072	**14.85**	2.03860	**22.63**	4.72877	**48.03**
10,000	5,000	.31392	**59.28**	.58098	**72.12**	1.16265	**115.03**	1.84178	**280.07**
10,000	10,000	.23267	**114.89**	.41504	**140.94**	0.75079	**232.98**	0.98552	**610.28**
10,000	20,000	.15575	**221.03**	.26751	**273.81**	0.43622	**473.10**	0.49935	**1301.84**
10,000	40,000	.09549	**423.61**	.15870	**530.13**	0.23561	**961.07**	0.24998	**2700.17**
10,000	80,000	.05481	**812.31**	.08882	**1025.89**	0.12204	**1948.44**	0.12500	**5500.01**
20,000	100	.49690	0.62	.99256	0.75	2.47218	1.13	6.84541	2.26
20,000	1,000	.47079	**6.21**	.93058	**7.46**	2.24649	**11.28**	5.67750	**23.29**
20,000	5,000	.38372	**30.30**	.73176	**36.66**	1.59247	**56.99**	3.05482	**129.15**
20,000	10,000	.31392	**59.28**	.58098	**72.12**	1.16265	**115.03**	1.84178	**280.07**
20,000	20,000	.23267	**114.89**	.41504	**140.94**	0.75079	**232.98**	0.98552	**610.28**
20,000	40,000	.15575	**221.03**	.26751	**273.81**	0.43622	**473.10**	0.49935	**1301.84**
20,000	80,000	.09549	**423.61**	.15870	**530.13**	0.23561	**961.07**	0.24998	**2700.17**
40,000	100	.49844	0.31	.99627	0.37	2.48601	0.56	6.92198	1.13
40,000	1,000	.48490	3.11	.96395	3.74	2.36666	**5.63**	6.28066	**11.45**
40,000	5,000	.43341	**15.36**	.84376	**18.52**	1.94813	**28.33**	4.35109	**60.88**
40,000	10,000	.38372	**30.30**	.73176	**36.66**	1.59247	**56.99**	3.05482	**129.15**
40,000	20,000	.31392	**59.28**	.58098	**72.12**	1.16265	**115.03**	1.84178	**280.07**
40,000	40,000	.23267	**114.89**	.41504	**140.94**	0.75079	**232.98**	0.98552	**610.28**
40,000	80,000	.15575	**221.03**	.26751	**273.81**	0.43622	**473.10**	0.49935	**1301.84**
80,000	100	.49922	0.16	.99813	0.19	2.49299	0.28	6.96081	.56
80,000	1,000	.49232	1.56	.98162	1.87	2.43156	2.81	6.62399	**5.68**
80,000	5,000	.46407	**7.74**	.91480	**9.31**	2.19075	**14.12**	5.41165	**29.35**
80,000	10,000	.43341	**15.36**	.84376	**18.52**	1.94813	**28.33**	4.35109	**60.88**
80,000	20,000	.38372	**30.30**	.73176	**36.66**	1.59247	**56.99**	3.05482	**129.15**
80,000	40,000	.31392	**59.28**	.58098	**72.12**	1.16265	**115.03**	1.84178	**280.07**
80,000	80,000	.23267	**114.89**	.41504	**140.94**	0.75079	**232.98**	0.98552	**610.28**

[a] Errors exceeding five percent of the target value are shown in bold font.

Table 3. Computational Errors from Higher-Order Risk Preferences with Logarithmic Utility[a]

		Risk aversion		Prudence		Temperance		Edginess	
x	***ε***	***R*(*x*, ε)**	**%Error**	***P*(*x*, ε)**	**%Error**	***T*(*x*, ε)**	**%Error**	***G*(*x*, ε)**	**%Error**
10,000	100	.98524	1.50	1.95127	2.50	2.89864	3.50	3.82792	4.50
10,000	1,000	.87072	**14.85**	1.60280	**24.78**	2.22684	**34.72**	2.76505	**44.66**
10,000	5,000	.58098	**72.12**	0.90411	**121.21**	1.10915	**170.48**	1.25051	**219.87**
10,000	10,000	.41504	**140.94**	0.59058	**238.65**	0.68661	**336.93**	0.74688	**435.56**
10,000	20,000	.26751	**273.81**	0.35169	**468.69**	0.39213	**665.06**	0.41568	**862.29**
10,000	40,000	.15870	**530.13**	0.19615	**919.63**	0.21239	**1312.52**	0.22133	**1707.25**
10,000	80,000	.08882	**1025.89**	0.10496	**1805.53**	0.11143	**2592.23**	0.11486	**3382.61**
20,000	100	.99256	0.75	1.97532	1.25	2.94843	1.75	3.91235	2.24
20,000	1,000	.93058	**7.46**	1.77870	**12.44**	2.55484	**17.42**	3.26778	**22.41**
20,000	5,000	.73176	**36.66**	1.23964	**61.34**	1.61236	**86.06**	1.89733	**110.82**
20,000	10,000	.58098	**72.12**	0.90411	**121.21**	1.10915	**170.48**	1.25051	**219.87**
20,000	20,000	.41504	**140.94**	0.59058	**238.65**	0.68661	**336.93**	0.74688	**435.56**
20,000	40,000	.26751	**273.81**	0.35169	**468.69**	0.39213	**665.06**	0.41568	**862.29**
20,000	80,000	.15870	**530.13**	0.19615	**919.63**	0.21239	**1312.52**	0.22133	**1707.25**
40,000	100	.99627	0.37	1.98758	0.62	2.97396	0.88	3.93700	1.60
40,000	1,000	.96395	3.74	1.88262	**6.23**	2.75912	**8.73**	3.59630	**11.23**
40,000	5,000	.84376	**18.52**	1.52767	**30.92**	2.09310	**43.33**	2.56829	**55.75**
40,000	10,000	.73176	**36.66**	1.23964	**61.34**	1.61236	**86.06**	1.89733	**110.82**
40,000	20,000	.58098	**72.12**	0.90411	**121.21**	1.10915	**170.48**	1.25051	**219.87**
40,000	40,000	.41504	**140.94**	0.59058	**238.65**	0.68661	**336.93**	0.74688	**435.56**
40,000	80,000	.26751	**273.81**	0.35169	**468.69**	0.39213	**665.06**	0.41568	**862.29**
80,000	100	.99813	0.19	1.99377	0.31	2.98751	0.42	4.11462	-2.79
80,000	1,000	.98162	1.87	1.93947	3.12	2.87439	4.37	3.78720	**5.62**
80,000	5,000	.91480	**9.31**	1.73107	**15.54**	2.46389	**21.76**	3.12540	**27.98**
80,000	10,000	.84376	**18.52**	1.52767	**30.92**	2.09310	**43.33**	2.56829	**55.75**
80,000	20,000	.73176	**36.66**	1.23964	**61.34**	1.61236	**86.06**	1.89733	**110.82**
80,000	40,000	.58098	**72.12**	0.90411	**121.21**	1.10915	**170.48**	1.25051	**219.87**
80,000	80,000	.41504	**140.94**	0.59058	**238.65**	0.68661	**336.93**	0.74688	**435.56**

[a] For logarithmic utility, the application of differential calculus yields $r(x) = 1$, $p(x) = 2$, $t(x) = 3$, and $g(x) = 4$. Errors exceeding five percent of the target value are shown in bold font.

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Figure 1. The Slope of Utility at a Point versus the Slope Between Two Points