1. Write the **formal definition** of the *intersection* of two sets.

Given \( A = \{1, 3, 4, 5\} \), \( B = \{3, 5, 6, 7\} \), and \( C = \{4, 6, 7, 8\} \) find the set \( (A \cap C) \cup (B \cap C) \).

2. Write the **formal definition** of the *Cartesian Product* of two sets.

Given \( A = \{a, b, c, d\} \) and \( B = \{b, d\} \), calculate the following:

   (a) \( B \times (A - B) \)  
   (b) \( |\mathcal{P}(A - B) \times \mathcal{P}(B - A)| \)

3. State and prove any one of the De Morgan’s Laws in logic.

4. For any two given statements \( P \) and \( Q \), prove that \( P \Rightarrow Q \) is logically equivalent to \( (\sim P) \lor Q \).

5. Consider the set defined by \( S = \{x \in \mathbb{R} \mid 2x + 5 < 0 \text{ and } \sqrt{1-x-x} = 5\} \). Find \( S \) (In other words, find all elements of \( S \), and write the answer in the proper notation). Make sure to do this problem completely by hand.

6. Consider the open sentences given by \( P(x, y) : |x| + |y| > 2 \), \( Q(x, y) : y^2 - x^2 \leq 4 \) where \( x, y \in \mathbb{R} \).

   (a) For what values of \((x, y)\) in the set \( \{(1,2), (-1,3)\} \) the implication \( P(x, y) \Rightarrow Q(x, y) \) is true?

   (b) Graphically illustrate all \( (x, y) \in \mathbb{R} \times \mathbb{R} \) such that \( P(x, y) \land Q(x, y) \) is true.

7. Consider the real function given by \( f(x) = (x - 1)^3(2x + 3) \) where \( x \) is any real number.

Let \( A = \{x|x \in \mathbb{R}, \text{and } x \text{ is a critical number of } f\} \) and \( B = \{x|x \in \mathbb{R} \text{ and } f \text{ has a local extrema at } x\} \). Find \( A - B \). Show all the work by hand.

8. Prove the following using **formal writing**:

\[ \forall \theta \in \mathbb{R} \ (\sin \theta < \frac{1}{3} \Rightarrow 4\sin \theta + \cos \theta \leq 1 + \sqrt{2}) \]
1. **Defn (Intersection of two sets):**
   For any two sets \( A \) and \( B \), the intersection of \( A \) and \( B \), denoted by \( A \cap B \), is defined as:
   \[ A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \]

   Given: \( A = \{3, 4, 5\} \), \( B = \{3, 5, 7\} \), and \( C = \{4, 6, 7, 8\} \).

   Then, by defn, \( A \cap C = \{4\} \) and \( B \cap C = \{6, 7\} \).

   \[ \therefore \text{by defn of union of 2 sets, } (A \cap C) \cup (B \cap C) = \{4\} \cup \{6, 7\} = \{4, 6, 7\} \]

2. **Defn (Cartesian Product of Two Sets):**
   For any two sets \( A \) and \( B \), the Cartesian Product of \( A \) and \( B \), denoted by \( A \times B \), is defined as:
   \[ A \times B = \{(a, b) \mid a \in A, b \in B\} \]

   Given: \( A = \{a, b, c, d\} \) and \( B = \{e, f\} \).

   \( a) \quad B \times (A-B) = \{e, f\} \times \{a, c\} = \{(c, a), (c, b), (d, a), (d, c)\} \)

   \( b) \quad |P(A-B) \times P(B-A)|\]

   First: \( A-B = \{a, c\} \) and \( B-A = \emptyset \).

   \[ \therefore |A-B| = 2 \quad \text{and} \quad |B-A| = 0 \]

   \[ \therefore \text{By thm, } |P(A-B)| = 2^{|A-B|} = 2^2 = 4 \quad \text{and similarly,} \]

   \[ |P(B-A)| = 2^{|B-A|} = 2^0 = 1 \]

   Then again by thm, \( |P(A-B) \times P(B-A)| = |P(A-B)| \cdot |P(B-A)| = 4 \cdot 1 = 4 \)

   \[ \therefore |P(A-B) \times P(B-A)| = 4 \]
3) De Morgan's Laws (of Logic):

For any 2 statements $P$ and $Q$

(a) $\sim (P \land Q) \equiv (\sim P) \lor (\sim Q)$

(b) $\sim (P \lor Q) \equiv (\sim P) \land (\sim Q)$.

Proof of (a) using truth tables:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$\sim P$</th>
<th>$\sim Q$</th>
<th>$(\sim P) \lor (\sim Q)$</th>
<th>$P \land Q$</th>
<th>$\sim (P \land Q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
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<tr>
<td>F</td>
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<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

These share the same truth values

$\therefore \sim (P \land Q) \equiv (\sim P) \lor (\sim Q)$.

4) To prove: $(P \Rightarrow Q) \equiv (\sim P) \lor Q$

proof:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$\sim P$</th>
<th>$(\sim P) \lor Q$</th>
<th>$P \Rightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
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<td>F</td>
<td>F</td>
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<td>T</td>
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</tr>
</tbody>
</table>

These two share the same truth values

$\therefore (P \Rightarrow Q) \equiv (\sim P) \lor Q$.
5. Let $S = \{ x \in \mathbb{R} \mid 2x + 3 < 0 \text{ and } \sqrt{1-x} - x = 5 \}$

Find $S$.

Suppose $x \in S$.

This means the same as (i.e. if and only if)

$x \in \mathbb{R}$ and $2x + 3 < 0$ and $\sqrt{1-x} - x = 5$.

Now let's solve $\sqrt{1-x} - x = 5$ \(\text{\textcircled{1}}\)

\[ \therefore \sqrt{1-x} = x + 5 \]

Now, square both sides. However, note that the squared equation is NOT EQUIVALENT to the original equation, as we might get extraneous solutions. Thus, we must check our final answers on the original eqn \(\text{\textcircled{1}}\)

\[ (\sqrt{1-x})^2 = (x+5)^2 \]

\[ 1-x = x^2 + 10x + 25 \]

\[ 0 = x^2 + 11x + 24 \]

\[ 0 = (x+3)(x+8) \]

\[ \therefore x = -3 \text{ or } x = -8 \]

However, we MUST check these answers back in \(\text{\textcircled{1}}\).

$x = -3$ satisfies \(\text{\textcircled{1}}\), but $x = -8$ does not satisfy \(\text{\textcircled{1}}\) (i.e. $x = -8$ is an extraneous solution).

The only solution for \(\text{\textcircled{1}}\) is $x = -3$.

Also, going back to \(\text{\textcircled{1}}\), $-3 \in \mathbb{R}$ and $2(-3) + 5 = -1 < 0$.

\[ \therefore S = \{-3\} \]

P.S. The wrong answer is $S = \{-3, -8\}$. 

Given: \( P(x, y) : |x| + |y| > 2 \) and \( Q(x, y) : y^2 - x^2 \leq 4 \) where \( x, y \in \mathbb{R} \).

(a) When \((x, y) = (1, 2)\), \( P(1, 2) \) is true because \(|1| + |2| = 3 > 2\) and \( Q(1, 2) \) is true because \((2)^2 - (1)^2 = 3 \leq 4\).

\( \therefore P(1, 2) \Rightarrow Q(1, 2) \) is true.

Now, when \((x, y) = (-1, 3)\): \( P(-1, 3) \) is true, because \(|-1| + |3| = 4 > 2\)

\( Q(-1, 3) \) is false because \((3)^2 - (-1)^2 = 8 \neq 4\).

\( \therefore P(-1, 3) \Rightarrow Q(-1, 3) \) is false.

Thus, the only values in the set \( \{(1, 2), (-1, 3)\}\) for which \( P(x, y) \Rightarrow Q(x, y) \) is true is \((1, 2)\).

(b) \( P(x, y) \land Q(x, y) \) is true iff \( P(x, y) \) is true and \( Q(x, y) \) is true

i.e. iff \(|x| + |y| > 2\) and \( y^2 - x^2 \leq 4\)

and

So, the final answer is the overlap of the two shaded regions:
\[ f(x) = (x - 1)^3 (2x + 3) \]

**Domain of** \( f = \mathbb{R} \).  

The critical points of \( f \) are the points on the domain of \( f \) at which the derivative \( f'(x) \) is zero or does not exist.

By product rule: 
\[
\begin{align*}
  f''(x) &= (x - 1)^3 \cdot 2 + 3(x - 1)^2 (2x + 3) \\
  &= (x - 1)^2 \left[ 2(x - 1) + 3(2x + 3) \right] \\
  &= (x - 1)^2 (8x + 7).
\end{align*}
\]

\[ f'(x) = (x - 1)^2 (8x + 7) \]

\[
\begin{aligned}
  &\text{Zero} \\
  &\text{B N E} \\
  &\text{Never}
\end{aligned}
\]

\[ x = 1, \text{ or } x = -\frac{7}{8} \]

Both are in the domain of \( f \).

\[ A = \left\{ 1, -\frac{7}{8} \right\} \]

Now, check which one of these critical points correspond to local extrema by using First Derivative Test.

Sign of \( f'(x) \) :  
\[
\begin{array}{c|c|c|c}
\text{Sign} & - & + & + \\
\hline
x & -\frac{7}{8} & 1
\end{array}
\]

\[ x + \text{ at } x = -7/8, f(x) \text{ has a local min} \]

but at \( x = 1 \), \( f(x) \) has neither min nor max.

\[ B = \left\{ -\frac{7}{8} \right\} \]

Now, we have \( A = \left\{ 1, -\frac{7}{8} \right\} \) and \( B = \left\{ -\frac{7}{8} \right\} \).

\[ A - B = \left\{ 1 \right\}. \]
Prove:

Let \( \theta \in \mathbb{R} \) such that \( \sin \theta < \frac{1}{3} \) \( \Rightarrow \) \( 4 \sin \theta + \cos \theta \leq 1 + \sqrt{2} \)

**Pf:** Let \( \theta \in \mathbb{R} \) be arbitrary.

Suppose \( \sin \theta < \frac{1}{3} \)

[we want to show: \( 4 \sin \theta + \cos \theta \leq 1 + \sqrt{2} \)]

By (1), \( 3 \sin \theta < 1 \)

Now \( \sin \theta + \cos \theta = \sqrt{2} \left( \frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta \right) \)

\[ = \sqrt{2} \left( \cos \frac{\pi}{4} \cos \theta + \sin \frac{\pi}{4} \sin \theta \right) = \sqrt{2} \sin \left( \frac{\pi}{4} + \theta \right) \]

\[ \Rightarrow \sin \theta + \cos \theta = \sqrt{2} \sin \left( \frac{\pi}{4} + \theta \right) \leq \sqrt{2} \] because for any \( \theta \in \mathbb{R} \)

Now, add lines (2) and (3):

\[ 3 \sin \theta + (\sin \theta + \cos \theta) \leq 1 + \sqrt{2} \]

\[ \Rightarrow 4 \sin \theta + \cos \theta \leq 1 + \sqrt{2} \]

This is what we wanted to prove.

Q.E.D.