1. Write the formal definition of the union of two sets.
Given $A = \{1, 3, 4, 5\}, B = \{3, 5, 6, 7\}, \text{and } C = \{4, 6, 7, 8\}$ find the set $A \cup (B \cap C)$.

2. Write the formal definition of the Cartesian Product of two sets.
Given $A = \{a, b, c, d\}$ and $B = \{b, d, e\}$, calculate the following:
(a) $A \times (B - A)$
(b) $|\mathcal{P}(A) \times B|$

3. Consider the set defined by $S = \{x \mid x \in \mathbb{C} - \mathbb{Q} \text{ and } x^5 - 2x^3 + 8x^2 - 16 = 0\}$. Find $S$ (In other words, find all elements of $S$, and write the answer in the proper notation). Make sure to do this problem completely by hand.

4. Construct a truth table for the statement $(P \land Q) \Rightarrow \neg P$

5. Prove that any implication is logically equivalent to its contrapositive.

6. Consider the open sentences given by $P(x, y): x^2 - y^2 > 0$, $Q(x, y): x^2 - y^2 \leq 1$ where $x, y \in \mathbb{R}$.
(a) For what values of $(x, y)$ in the set $\{(1,2), (3,-1), (-2,1)\}$ the implication $P(x, y) \Rightarrow Q(x, y)$ is true?
(b) Graphically illustrate all $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $P(x, y) \land Q(x, y)$ is true.

7. Consider $S \subseteq \mathbb{R} \times \mathbb{R}$ defined by $S = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } 4x(1-x) < y^2 - 3\}$ . Graphically represent this set $S$ using a BIG and CLEAR graph. Make sure to do this problem by hand.

8. Prove the following using formal writing:
$$\forall \theta \in \mathbb{R} \left( \sin \theta < \frac{1}{2} \Rightarrow 4 - 5 \sin \theta > 2 \cos^2 \theta \right)$$
1) Defn (Union of two sets):

For any two sets \( A \) and \( B \), their union denoted by \( A \cup B \), is defined as

\[
A \cup B = \{ x | x \in A \text{ or } x \in B \}
\]

Given \( A = \{1, 3, 4, 5\} \) and \( B = \{3, 5, 6, 7\} \) and \( C = \{4, 6, 7, 8\} \)

By defn, \( B \cap C = \{6, 7\} \).

Then, again by defn, \( A \cup (B \cap C) = \{1, 3, 4, 5\} \cup \{6, 7\} = \{1, 3, 4, 5, 6, 7\} \)

\[
\therefore \quad A \cup (B \cap C) = \{1, 3, 4, 5, 6, 7\}
\]

2) Defn (Cartesian Product of two sets):

For any two sets \( A \) and \( B \), their Cartesian Product denoted by \( A \times B \) is defined as

\[
A \times B = \{ (a, b) | a \in A \text{ and } b \in B \}
\]

Given: \( A = \{a, b, c, d\} \) and \( B = \{b, d, e\} \).

By defn, \( B - A = \{x | x \in B \text{ and } x \notin A\} = \{e\} \)

Again, by defn, \( A \times (B - A) = \{a, b, c, d\} \times \{e\} \)

\[
= \{(a, e), (b, e), (c, e), (d, e)\}
\]

By thm, \( |P(A) \times B| = |P(A)| \cdot |B| \) \( \tag{1} \)

Also by thm, \( |P(A)| = 2^{\left|A\right|} \) \( \tag{2} \)

\[
\therefore \quad \left|P(A) \times B\right| = 2^{\left|A\right|} \cdot |B| = 2^4 \cdot (3) = (16)(3) = 48
\]

\[
\therefore \quad \left|P(A) \times B\right| = 48
\]
3) Let \( S = \{ x \mid x \in \mathbb{C} - \mathbb{Q} \text{ and } x^5 - 2x^3 + 8x^2 - 16 = 0 \} \).

First, let's find all real or complex solutions of

\[ x^5 - 2x^3 + 8x^2 - 16 = 0. \]  \( \text{---} (1) \)

By Fundamental Theorem of Algebra, the above equation has exactly 5 real or complex solutions.

Let's factor the left-hand side of eqn \( (1) \):

\[ x^5 - 2x^3 + 8x^2 - 16 = x^3(x^2 - 2) + 8(x^2 - 2) = (x^2 - 2)(x^3 + 8) = (x^2 - 2)(x + 2)(x^2 - 2x + 4) \]

\[ \therefore \text{ The eqn } (1) \text{ is equivalent to:} \]

\[ (x^2 - 2)(x + 2)(x^2 - 2x + 4) = 0 \]

\[ \therefore x^2 - 2 = 0 \text{ or } x + 2 = 0 \text{ or } x^2 - 2x + 4 = 0 \]

\[ \therefore x = \pm \sqrt{2} \text{ or } x = -2 \text{ or } x = \frac{2 \pm \sqrt{4-16}}{2} = 1 \pm \sqrt{3}i \]

Thus the 5 solutions of \( (1) \) are:

\[ x = -\sqrt{2}, \sqrt{2}, 2, 1 - \sqrt{3}i \text{ and } 1 + \sqrt{3}i \]

However, the only rational solution of above is 2.

But we are only looking for solutions \( x \) at \( x \in \mathbb{C} - \mathbb{Q} \)

\[ \therefore \text{ By defn of } S, \quad S = \{-\sqrt{2}, \sqrt{2}, 1 - \sqrt{3}i, 1 + \sqrt{3}i\}. \]
4. Truth Table for \((P \land \theta) \Rightarrow \sim P\)

<table>
<thead>
<tr>
<th></th>
<th>α</th>
<th>(P \land \alpha)</th>
<th>\sim P</th>
<th>((P \land \alpha) \Rightarrow \sim P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

5. Consider the implication \(P \Rightarrow \alpha\) where \(P\) and \(\alpha\) are arbitrary statements.

By defn, its contrapositive is \(\sim \alpha \Rightarrow \sim P\).

We want to show that \((P \Rightarrow \alpha) \equiv (\sim \alpha \Rightarrow \sim P)\).

So, we construct truth tables for \(P \Rightarrow \alpha\) and \(\sim \alpha \Rightarrow \sim P\).

<table>
<thead>
<tr>
<th></th>
<th>α</th>
<th>\sim \alpha</th>
<th>\sim P</th>
<th>(P \Rightarrow \alpha)</th>
<th>(\sim \alpha \Rightarrow \sim P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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<td>F</td>
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<td>T</td>
<td>T</td>
<td>T</td>
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</tbody>
</table>

By inspecting the last two columns, we see that both \(P \Rightarrow \alpha\) and \(\sim \alpha \Rightarrow \sim P\) share the same truth values.

Hence the assertion follows.
6) Let \( P(x, y) : x^2 - y^2 > 0 \)
\( Q(x, y) : x^2 - y^2 \leq 1 \) \( \text{where } x, y \in \mathbb{R} \).

(a) Let \( S = \{(1, 2), (3, -1), (-3, 1)\} \).

For \((x, y) = (1, 2)\):
\( P(1, 2) : 1^2 - 2^2 > 0 \) is false since \( 1^2 - 2^2 = -3 \) and \(-3 < 0\).
\( Q(1, 2) : 1^2 - 2^2 \leq -1 \) is true since \(-3 \leq -1\).
\( \therefore \) By definition of implication, \( P(1, 2) \implies Q(1, 2) \) is true.

For \((x, y) = (3, -1)\):
\( P(3, -1) : 3^2 - (-1)^2 > 0 \) is true since \( 8 > 0 \).
\( Q(3, -1) : 3^2 - (-1)^2 \leq 1 \) is false since \( 8 > 1 \).
\( \therefore \) \( P(3, -1) \implies Q(3, -1) \) is false.

For \((x, y) = (-3, 1)\):
\( P(-3, 1) : (-3)^2 - 1^2 > 0 \) is true since \( 8 > 0 \).
\( Q(-3, 1) : (-3)^2 - 1^2 \leq 1 \) is false since \( 8 > 1 \).
\( \therefore \) \( P(-3, 1) \implies Q(-3, 1) \) is false.

\( \therefore \) The only \((x, y) \in S\) for which \( P(x, y) \implies Q(x, y) \) is true is \( (1, 2) \).

(b) \( P(x, y) \land Q(x, y) \) is true iff both \( P(x, y) \) and \( Q(x, y) \) are true.

Thus we need \((x, y) \in \mathbb{R} \times \mathbb{R}\) st \( x^2 - y^2 > 0 \) and \( x^2 - y^2 \leq 1 \).

The graph of \( x^2 - y^2 = 0 \) is a pair of parallel lines.
The graph of \( x^2 - y^2 = 1 \) is a hyperbola.
The required region is the intersection (overlap) of the above regions:

Let \( S = \{(x,y) \mid x, y \in \mathbb{R} \text{ and } 4x(1-x) \leq y^2 - 3\} \).

Suppose \( x, y \in \mathbb{R} \) are such that \( 4x(1-x) \leq y^2 - 3 \).

To determine the region of \( \mathbb{R}^2 \) satisfied by this inequality, let's first graph the curve given by the equality

\[
4x(1-x) = y^2 - 3
\]

\[
0 = 4x^2 - 4x + y^2 - 3
\]

\[
(4x^2 - 4x) + y^2 = 3 \quad \text{Now complete the square}
\]

\[
4(x^2 - x) + y^2 = 3
\]

\[
4(x^2 - x + \left(\frac{1}{2}\right)^2) + y^2 = 3 + 4\left(\frac{1}{2}\right)^2
\]

\[
4\left(x - \frac{1}{2}\right)^2 + y^2 = 4
\]

\[
\frac{(x - \frac{1}{2})^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{\left(\frac{2}{2}\right)^2} = 1
\]

which is the equation of an ellipse centered at \( \left(\frac{1}{2}, 0\right) \).

Now the test pt \( \left(\frac{1}{2}, 0\right) \) does not satisfy the inequality 1.

So we have to shade the outside of the ellipse, excluding the boundary.

The set \( S \) is shaded in blue.
8) Prove that:
\[ \forall \theta \in \mathbb{R} \left( \sin \theta < \frac{1}{2} \implies 4 - 5 \sin \theta > 2 \cos^2 \theta \right) \]

Proof: Let \( \theta \in \mathbb{R} \) be arbitrary.

Suppose \( \sin \theta < \frac{1}{2} \)

\[ : 2 \sin \theta - 1 < 0 \quad (1) \]

Also, we know that \( \sin \theta - 2 < 0 \quad (2) \)

By (1) and (2), \( (2 \sin \theta - 1)(\sin \theta - 2) > 0 \)

\[ : 2 \sin^2 \theta - 5 \sin \theta + 2 > 0 \]

\[ : (2(1 - \cos^2 \theta)) - 5 \sin \theta + 2 > 0 \]

\[ \therefore 4 - 5 \sin \theta > 2 \cos^2 \theta \]

Thus we have proved that

\[ \forall \theta \in \mathbb{R} \left( \sin \theta < \frac{1}{2} \implies 4 - 5 \sin \theta > 2 \cos^2 \theta \right) \]

Q. E. D.