1. Find the exact value of \( \lim_{x \to 3} \frac{2x^2 + x - 21}{9 - x^2} \). Show your work by hand.

2. Find the exact value of \( \lim_{x \to 2} \frac{16 - 2x^3}{\sqrt{x} + 2 - 2} \). Show your work by hand.

3. Find the exact value of the \( \lim_{x \to 0} \frac{\cos^3 3x - 1}{\cos(4x) - 1} \). Show your work by hand.

4. Draw a BIG, CLEAR graph of \( f(x) = \begin{cases} 3x - 2 & \text{if } x < 2 \\ 2x^2 - 4 & \text{if } x \geq 2 \end{cases} \)

Is \( f \) continuous at \( x = 2 \)? Give the precise mathematical reason using the definition of continuity \( \text{without} \) relating to the graph. If discontinuous, name the type of discontinuity.

5. Given \( f(x) = -2x + 3x^3 + 6 \), calculate \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) by hand. Simplify your answer. Show your work carefully.

6. Write the definition of vertical asymptote in precise mathematical terms.

Find all the vertical asymptotes of \( f(x) = x \csc(2x) \) for \( -\pi \leq x \leq 2\pi \). Make sure to show the necessary limit calculations.

7. Write the definition of the continuity of a function on a closed interval, in precise mathematical terms. Draw the typical graph of \( \text{any} \) such function, continuous on a closed interval. Make sure to draw a big and clear graph with all the necessary components.

8. Given \( f(x) = \frac{1 - x}{2 + \sqrt{x}} \), calculate \( \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \) by hand. Simplify your answer. Show your work carefully.


Consider the function \( f(x) = 4 - \cos(2x) - 5\sin(x) \) on the interval \([-\pi, 2\pi]\). Find the exact \( c \)-values satisfying the conditions of the IVT with \( f(c) = 6 \).
1. \[ \lim_{x \to 3} \frac{2x^2 + x - 21}{x^2 - 9} \]
   \[= \lim_{x \to 3} \frac{(2x + 7)(x-3)}{(3-x)(3+x)} \]
   \[= \lim_{x \to 3} \frac{(2x+7)(x-3)}{(x-3)(3+x)} \]
   \[= \lim_{x \to 3} \frac{2x+7}{3+x} = -\frac{13}{6} \]

2. \[ \lim_{x \to 2} \frac{16 - 2x^3}{\sqrt{x^2 - 2}} \]
   \[= \lim_{x \to 2} \frac{2(8-x^3)}{\sqrt{x^2 - 2}} = \lim_{x \to 2} \frac{2(2-x)(4+2x+x^2)}{(\sqrt{x^2 - 2})(\sqrt{x^2 + 2})} \]
   \[= \lim_{x \to 2} \frac{-2(2-x)(4+2x+x^2)}{(x+2)-4} \]
   \[= \lim_{x \to 2} \left[ -2(4+2x+x^2)(\sqrt{x^2 + 2}) \right] = -2(12)(4) = -96 \]

3. \[ \lim_{x \to 0} \frac{\cos^3(3x) - 1}{\cos(4x) - 1} \]
   \[= \lim_{x \to 0} \frac{\cos(3x-1)(\cos(3x+1)(\cos^2(3x) + \cos(3x+1))}{\sqrt{2 \sin^2(2x)} - x} \]
   \[= \lim_{x \to 0} \left[ \frac{\cos(3x-1)(\cos(3x+1)(\cos^2(3x) + \cos(3x+1))}{\sqrt{2 \sin^2(2x)}(\cos(3x+1))} \right] \]
   \[= \frac{3}{4} \lim_{x \to 0} \left[ \frac{\sin(3x) \cdot \cos(3x) + 9x^2}{\sin(2x) \cdot \cos(2x)} \right] = \frac{3}{4} \frac{(1)(1)}{4} = \frac{27}{16} \]

4. Graph: \[ f(x) = \begin{cases} 
3x-2 & \text{if } x < 2 \\
2x^2 - 4 & \text{if } x \geq 2 
\end{cases} \]

Yes, \( f(x) \) is continuous at \( x = 2 \)

Math. Reason: 
(i) \( f(2) \) exists and is equal to 4

and (ii) \( \lim_{x \to 2^-} f(x) = 4 \) because \( \lim_{x \to 2^-} f(x) = 3(2) - 2 = 4 \)

and \( \lim_{x \to 2^+} f(x) = 2(2)^2 - 4 = 4 \)

and these two are equal

and (iii) \( f(2) = \lim_{x \to 2} f(x) \).
5. Given \( f(x) = -2x + 3x^3 + 6 \)

\[
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-2(x+h) + 3(x+h)^3 + 6}{h} - \lim_{h \to 0} \frac{-2x + 3x^3 + 6}{h} \\
= \lim_{h \to 0} \frac{-2x - 2h + 3x^3 + 9x^2h + 9xh^2 + 3h^3 + 6}{h} - \frac{2x - 3x^3 - 6}{h} \\
= \lim_{h \to 0} \frac{(-2 + 9x^2 + 3xh + 3h^2)h}{h} = -2 + 9x^2 + 9x(0) + 3(0)^2 = -2 + 9x^2 \\
\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = -2 + 9x^2
\]

6. Defn (Vertical Asymptote of a Function): A vertical line \( x = a \) is called a vertical asymptote of a function \( f(x) \) if any one of the following conditions are satisfied:

(i) \( \lim_{x \to a^-} f(x) = -\infty \) or (ii) \( \lim_{x \to a} f(x) = +\infty \) or (iii) \( \lim_{x \to a^+} f(x) = -\infty \) or (iv) \( \lim_{x \to a^+} f(x) = +\infty \)

Find the V.A's of \( f(x) = x \csc(2x) \) for \( -\pi \leq x \leq 2\pi \)

\( f(x) = \frac{x}{\sin(2x)} \)

Candidates for V.A: Set \( \sin(2x) = 0 \).

\( 2x = \ldots \ldots -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots \)

\( x = \ldots \ldots -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, \ldots \)

Since \( -\pi \leq x \leq 2\pi \), the ONLY candidates are: \( x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \)

Now check the candidates:

\( x = 0: \lim_{x \to 0^-} \frac{x}{\sin(2x)} = \lim_{x \to 0^-} \frac{1}{2x} = \frac{1}{2} \) and also \( \lim_{x \to 0^+} \frac{x}{\sin(2x)} = \frac{1}{2} \)

\( \therefore x = 0 \text{ is not a V.A} \)

\( x = -\pi: \lim_{x \to -\pi^-} \frac{x}{\sin(2x)} = +\infty \) (show details) \( \therefore x = -\pi \text{ is a V.A} \)

Similarly we can show that each of \( x = -\frac{\pi}{2}, \frac{\pi}{2}, \pi, 3\frac{\pi}{2}, 2\pi \) are V.A's.

\( \therefore \text{The only V.A's are: } x = -\pi, -\frac{\pi}{2}, \frac{\pi}{2}, \pi, 3\frac{\pi}{2} \text{ and } 2\pi \).
7 Defn (Continuity on a closed interval):
Consider a function \( f(x) \) defined on a closed interval \([a, b]\). Then we say \( f(x) \) is continuous on \([a, b]\) if the following three conditions are satisfied:

(i) \( f(x) \) is continuous at any \( x \) where \( a < x < b \)

(ii) \( \lim_{x \to a^+} f(x) = f(a) \)

(iii) \( \lim_{x \to b^-} f(x) = f(b) \)

\[
\begin{align*}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(1-x-h) (2+\sqrt{x+h}) - (1-x)(2+\sqrt{x})}{(2+\sqrt{x+h})(2+\sqrt{x})} \\
&= \lim_{h \to 0} \frac{(1-x-h) (2+\sqrt{x+h}) - (1-x)(2+\sqrt{x+h})}{(2+\sqrt{x+h})(2+\sqrt{x})} \\
&= \lim_{h \to 0} \frac{(1-x)(2+\sqrt{x}) - h(2+\sqrt{x}) - (1-x)(2+\sqrt{x+h})}{h (2+\sqrt{x+h})(2+\sqrt{x})} \\
&= \lim_{h \to 0} \frac{(1-x)(2+\sqrt{x}) - (1-x)(2+\sqrt{x+h})}{h (2+\sqrt{x+h})(2+\sqrt{x})} \\
&= \lim_{h \to 0} \left\{ \frac{1}{h(2+\sqrt{x+h})(2+\sqrt{x})} \left[ \frac{1}{2+\sqrt{x+h}} - \frac{1}{2+\sqrt{x}} \right] \right\} \\
&= \lim_{h \to 0} \left\{ \frac{1}{h(2+\sqrt{x+h})(2+\sqrt{x})} \left[ \frac{1}{2+\sqrt{x+h}} - \frac{1}{2+\sqrt{x}} \right] \right\}
\end{align*}
\]
\[ \lim_{b \to 0} \left\{ \frac{-(1-x)}{(2+\sqrt{x+h})(2+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} - \frac{1}{(2+\sqrt{x+h})} \right\} \]

\[ = \frac{-(1-x)}{(2+\sqrt{x})(2+\sqrt{x})} - \frac{1}{(2+\sqrt{x})} = \frac{-(1-x)}{2\sqrt{x} (2+\sqrt{x})^2} - \frac{1}{(2+\sqrt{x})} \]

\[ = \frac{-(1-x) - 2\sqrt{x}(2+\sqrt{x})}{2\sqrt{x} (2+\sqrt{x})^2} = \frac{-1+\sqrt{x} - 4\sqrt{x} - 2\sqrt{x}}{2\sqrt{x} (2+\sqrt{x})^2} = \frac{-\sqrt{x} - 4\sqrt{x} - 1}{2\sqrt{x} (2+\sqrt{x})^2} \]

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9. **Thm (IVT):**

Suppose \( f(x) \) is a fn that is continuous on the closed interval \([a, b]\). Further suppose that \( k \) is any real number between \( f(a) \) and \( f(b) \). Then there exists a real number \( c \) in \([a, b]\) such that \( f(c) = k \).

**Given:** \( f(x) = 4 - \cos(2x) - 5\sin(x) \) on \([-\pi, \frac{3\pi}{2}]\)

Set \( f(c) = 6 \)

\[ 4 - \cos(2c) - 5\sin(c) = 6 \]

\[ 4 - (1 - 2\sin^2(c)) - 5\sin(c) = 6 \]

\[ 2\sin^2(c) - 5\sin(c) - 3 = 0 \]

\[ (2\sin(c) + 1)(\sin(c) - 3) = 0 \]

\[ \sin(c) = -\frac{1}{2} \text{ or } \sin(c) = 3 \]

\[ \sin(c) = -\frac{1}{2} \text{ or } \sin(c) = 3 \]

\[ S \text{ impossible!} \]

\[ c = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{7\pi}{6} \text{ (only 3 answers)} \]

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**END OF TEST**