1. Find the derivatives of the following functions. **Do not simplify** your answers.
   (a) \( y = 3\sqrt{1 - x^2} \)
   (b) \( f(x) = x\sin(2x) \)

2. Find the derivatives of the following functions. **Simplify** your answers.
   (a) \( y = \frac{4}{1 + x^2} \)
   (b) \( y = \sec^2(x^2) \)

3. Find the equation of the tangent line to the graph of \( f(x) = 2x^3 + x^2 - 5 \) at \( x = -1 \). Provide the answer in the slope-intercept form.

4. Find the \( x \)-coordinates of the points on the graph of \( f(x) = (2x + 3)^3(4 - 3x)^2 \) at which there is a horizontal tangent line. Provide the exact answers.

5. A ball is thrown vertically from the ground-level at an initial velocity of 60 ft/sec. Its height \( h \) (in feet) from the ground-level, at time \( t \) (in seconds) is given by the formula \( h(t) = -16t^2 + 60t \). Find the exact height of the ball when its velocity is equal to 30 ft/sec. Do this problem using principles of calculus, without using any equations from physics.

6. Find the exact **y-coordinates** of the points on the graph of \( f(x) = 5\cos(2x) + 4\sin(3x) \) where \( 0 \leq x \leq \pi \), at which the tangent line is perpendicular to the \( y \)-axis. Provide the exact answers by hand.

7. Find the slope of the tangent line(s) to the graph of \( f(x) = -x^3 + 2x + 1 \) which pass through the point \((2, 1)\). Provide the exact and the simplified answers.

8. Find \( \frac{dy}{dx} \) given that \( x\tan(xy^2) + 2y = x + 2 \) at \((\pi/4, 1)\). Provide the simplified answer.

9. Gas is pumped into a spherical balloon at the rate of \( 5 \text{ cm}^3/\text{min} \). Find how fast its radius is changing when the volume of the balloon is equal to \( 36\pi \text{ cm}^3 \). Provide the exact answer.
4 (a) \( y = 3 \sqrt{1-x^2} \) \( \therefore \) \( y = 3 (1-x^2)^{\frac{1}{2}} \) \( \therefore \) \( \frac{dy}{dx} = 3 \cdot \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \cdot (-2x) \)

(b) \( f(x) = x \sin(2x) \)
\( \therefore f'(x) = x \cdot \cos(2x) \cdot 2 + 1 \cdot \sin(2x) \)

2 (a) \( y = \frac{4}{1+x^2} = 4 (1+x^2)^{-1} \)
\( \therefore \frac{dy}{dx} = 4 (-1) \cdot (1+x^2)^{-2} \cdot 2x = -8x \quad \frac{1}{(1+x^2)^2} \)

(b) \( y = \sec^2(x^2) = [\sec(x^2)]^2 \)
\( \therefore \frac{dy}{dx} = 2 \left[ \sec(x^2) \right] \cdot \sec(x^2) \cdot \tan(x^2) \cdot 2x = 4x \sec^2(x^2) \tan(x^2) \)

3 \( f(x) = 2x^3 + x^2 - 5 \)
\( f'(x) = 6x^2 + 2x \)
\( \therefore m = f'(-1) = 6(-1)^2 + 2(-1) = 4 \)
Also: \( x_1 = -1 \) and \( y_1 = f(-1) = 2(-1)^3 + (-1)^2 - 5 = -2 + 1 - 5 = -6 \)
\( \therefore \) pt. on the graph is \( P(x_0, y_1) = (-1, -6) \)
The eqn. of tgt. line at \( P(x_1, y_1) \) is \( y - y_1 = m(x - x_1) \)
\( y + 6 = 4(x + 1) \) \( \therefore y = 4x - 2 \)

4 \( f(x) = (2x+3)^3(4-3x)^2 \)
\( \therefore f'(x) = (2x+3)^3 \cdot 2 \cdot (4-3x) (-3) + 3 (2x+3)^2 \cdot 2 \cdot (4-3x)^2 \)
\( = -6 (2x+3)^3 (4-3x) + 6 (2x+3)^2 (4-3x)^2 \)
\( = 6 (2x+3)^2 (4-3x) [-2(2x+3) + (4-3x)] \)
\( \therefore f'(x) = 6 (2x+3)^2 (4-3x) (-5x+1) \) \( \text{Set } f'(x) = 0 \) and solve for \( x \).
\( \therefore \) \( x = \frac{-3}{2} \) or \( x = \frac{4}{3} \) or \( x = \frac{1}{5} \)
\( h(t) = -16t^2 + 60t \)

\[ \text{velocity} = v(t) = h'(t) = -32t + 60. \]

Set \( v(t) = 30 \text{ ft/sec} \) and solve for \( t \)

\[ -32t + 60 = 30 \implies t = \frac{30}{32} = \frac{15}{16} \text{ sec}. \]

\[ h\left(\frac{15}{16}\right) = -16\left(\frac{15}{16}\right)^2 + 60\left(\frac{15}{16}\right) = \frac{675}{16} \text{ ft}. \]

\[ f(x) = 5\cos(2x) + 4\sin(3x) \quad \text{where} \quad 0 \leq x \leq \pi \]

\[ f'(x) = 5(-\sin 2x) \cdot 2 + 4\cos(3x) \cdot 3 \]

\[ = -10\sin 2x + 12\cos 3x \]

\[ f''(x) = -10\left[ 2\sin x \cos x \right] + 12\left[ 4\cos^3 x - 3 \cos x \right] \]

\[ = 4\cos x \left( -5\sin x + 12\cos^2 x - 9 \right) \]

\[ = 4\cos x \left[ -5\sin x + 12(1 - \sin^2 x) - 9 \right] \]

\[ = 4\cos x \left( -5\sin x + 12 - 12\sin^2 x - 9 \right) \]

\[ = 4\cos x \left( 12\sin^2 x + 5\sin x - 3 \right) = -4\cos x \left( 3\sin x - 1 \right) \left( 4\sin x + 3 \right) \]

\[ f'(x) = 0 \text{ gives that} \quad \cos x = 0 \text{ or} \quad \sin x = \frac{1}{3} \text{ or} \quad \sin x = -\frac{3}{4} \]

Not possible since \( 0 \leq x \leq \pi \)

\( \therefore \quad x = \frac{\pi}{2} \quad \text{or} \quad x \) is st \( \sin x = \frac{1}{3} \)

\[ \text{2 solutions for this range where} \quad 0 \leq x \leq \pi. \]

Now find \( y \) coordinate.

When \( x = \frac{\pi}{2} \):

\[ f\left( \frac{\pi}{2} \right) = 5\cos \left( \frac{\pi}{2} \right) + 4\sin \left( \frac{3\pi}{2} \right) = -5 - 4 = -9 \]

When \( x \) is st: \( \sin x = \frac{1}{3} \):

\[ f(x) = 5\cos(2x) + 4\sin(3x) \]

\[ f(x) = 5 \left[ 1 - 2\sin^2 x \right] + 4 \left[ 3 \left( \frac{1}{3} \right) - 4 \left( \frac{1}{3} \right)^3 \right] = \frac{197}{27} \]

\[ \therefore \text{The} \quad y \text{-coordinates at which the tangent line is horizontal are} \]

\[ y = -9 \quad \text{and} \quad y = \frac{197}{27}. \]
\[ f(x) = -x^3 + 2x + 1 \]
\[ f'(x) = -3x^2 + 2 \]

:. Slope of \( PQ = f'(t) = -3t^2 + 2 \) \hspace{1cm} (1)

Also Slope of \( PQ = \left( \frac{-t^3 + 2t + 1}{t - 2} \right) - 1 \) \hspace{1cm} (2)

By (1) and (2): \[-3t^2 + 2 = \frac{-t^3 + 2t}{t - 2} \]

\[ \therefore (-3t^2 + 2)(t - 2) = -t^3 + 2t \]

\[ -3t^3 + 6t^2 - 2t - 4 = -t^3 + 2t \]

\[ 0 = 2t^3 - 6t^2 + 4 \]

\[ \therefore 0 = t^3 - 3t^2 + 2. \]

\( t = 1 \) makes the R.H.S. zero, so \( t = 1 \) is a factor of \( t^3 - 3t^2 + 2 \).

To find other factors divide \( t^3 - 3t^2 + 2 \) by \( t - 1 \) either using Long Division (or Synthetic Division)

\[ (t - 1)(t^2 - 2t - 2) = 0 \]

\[ \therefore t = 1 \text{ or } t = 2 \pm \sqrt{4 + 8} = 2 \pm \sqrt{12} = 2 \pm \sqrt{4 \times 3} = 2 \pm 2\sqrt{3} \]

\[ \therefore t = 1 \text{ or } t = 1 + \sqrt{3} \text{ or } 1 - \sqrt{3} \]

When \( t = 1 \):
\[ f'(t) = -3t^2 + 2 = -3(1)^2 + 2 = -1 \]

When \( t = 1 + \sqrt{3} \):
\[ f'(t) = -3(1 + \sqrt{3})^2 + 2 = -3(4 + 2\sqrt{3}) + 2 = -10 + 6\sqrt{3} \]

When \( t = 1 - \sqrt{3} \):
\[ f'(t) = -3(1 - \sqrt{3})^2 + 2 = -3(4 - 2\sqrt{3}) + 2 = -10 - 6\sqrt{3} \]

\[ \therefore \text{Three slopes are:} \]
\[ m_1 = -1 \text{ and } m_2 = -10 + 6\sqrt{3} \text{ and } m_3 = -10 - 6\sqrt{3} \]
\( x \tan(xy^2) + 2y = x + 2 \)

Diff. both sides w.r.t. \( x \)

\[ x \sec^2(xy^2) \left[ x \cdot y \frac{dy}{dx} + 2y^2 \right] + \tan(xy^2) + 2 \frac{dy}{dx} = 1 \]

\[ 2x^2y \sec^2(xy^2) \frac{dy}{dx} + xy^2 \sec^2(xy^2) + \tan(xy^2) + 2 \frac{dy}{dx} = 1 \]

\[ \frac{dy}{dx} \left[ 2x^2y \sec^2(xy^2) + 2 \right] = 1 - xy^2 \sec^2(xy^2) - \tan(xy^2) \]

\[ \frac{dy}{dx} = \frac{1 - xy^2 \sec^2(xy^2) - \tan(xy^2)}{2x^2y \sec^2(xy^2) + 2} \]

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Let \( r = \) rad. of balloon

\( V = \) vol of balloon

\( t = \) time

Also know: \( V = \frac{4\pi r^3}{3} \)

So \( \frac{dv}{dr} = 4\pi r^2 \)

\( \frac{dv}{dt} \)

\[ \frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} \]

\[ 5 = 4\pi r^2 \frac{dr}{dt} \]

\[ \frac{dr}{dt} = \frac{5}{4\pi r^2} \]

\[ \frac{dr}{dt} \bigg|_{V=36\pi} = \frac{5}{4\pi (3)^2} = \frac{5}{36\pi} \text{ cm/min.} \]

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End of Test 2