IMPORTANT: Do not attempt to write or workout any answer on this piece of paper, as it will not be graded.

SHOW ALL WORK

1. Find the derivative of the given function. To save time DO NOT simplify the answers:

(a) \( y = \frac{4}{\sqrt{x^2 + 1}} \)

(b) \( y = \cot^4[(3x + 2)^3] \)

[Identical to handout/Done in class]

2. Find the equation of the tangent line to the graph of \( y = 3\sec^2(2x) \) at \( x = \pi/8 \). Leave the exact answer in the slope-intercept form.

[Identical to class]

3. Find \( \frac{dy}{dx} \) given that \( y = x \sin(x + 2y) \).

[Identical to class]

4. Find the exact x-coordinates of the points on the graph of \( y = (3x + 4)^2(3 - 2x)^3 \) at which the tangent line is horizontal.

[Identical to handout]

5. Given \( y = \frac{x}{\sqrt{4 - x^2}} \), find \( \frac{d^2y}{dx^2} \) and simplify the answer completely.

[Similar to handout]

6. Find the equation of the tangent lines the graph of \( f(x) = \sqrt{x^2 + 3} \) which pass through the point \( (3, 3) \). Provide the exact answers.

7. Find the rate of change of the surface area of a sphere with respect to its volume, when its diameter is equal to 4 inches. Provide the exact answer.

8. Find the exact critical points of \( f(x) = 6\sin x + 2\tan x - 5x \) where \( 0 \leq x < 2\pi \), \( x \neq \pi/2 \) and \( x \neq 3\pi/2 \).
1. (a) \[ y = \frac{4}{\sqrt{x^2+1}} \]
\[ \therefore \frac{dy}{dx} = 4 \left( \frac{1}{2} \right) (x^2+1)^{-\frac{3}{2}} \cdot 2x \]

(b) \[ y = \text{Cot}^4 (3x+2)^3 = [\text{Cot} (3x+2)^3]^4 \]
\[ \therefore \frac{dy}{dx} = 4 \left[ \text{Cot} (3x+2)^3 \right]^3 \cdot [-\text{Cosec}^2 (3x+2)^3] \cdot 3(3x+2)^2 \cdot 3 \]

2. \[ y = 3 \sec^2(2x) = 3 \left[ \sec(2x) \right]^2 \]
\[ \therefore y' = f'(x) = 3 \cdot 2 \left[ \sec(2x) \right] \cdot \sec(2x) \cdot \tan(2x) \cdot 2 \]
\[ \therefore f''(x) = 12 \sec^2(2x) \cdot \tan(2x) \]

Slope of tangent = \( m = f'\left( \frac{\pi}{8} \right) = 12 \sec^2\left( \frac{\pi}{4} \right) \tan \left( \frac{\pi}{4} \right) = 12 \left( \sqrt{2} \right)^2 (1) = 24 \)

Also \( \alpha = \frac{\pi}{8} \) and \( y_1 = f\left( \frac{\pi}{8} \right) = 3 \sec^2 \left( \frac{\pi}{4} \right) = 3 (\sqrt{2})^2 = 6 \)

\( \therefore \) pt. on the graph = \((x_1, y_1) = \left( \frac{\pi}{8}, 6 \right)\).

E. qn. of tangent line is \( y - y_1 = m(x - x_1) \)
\[ y - 6 = 24 \left( x - \frac{\pi}{8} \right) \]
\[ \therefore y = 24x + 6 - 3\pi \]

3. \[ y = x \sin(x+2y) \]
Diff. both sides w. r. t. \( x \)
\[ \therefore \frac{dy}{dx} = \frac{d}{dx} \left[ x \cdot \sin(x+2y) \right] \]
\[ \frac{dy}{dx} = x \cdot \cos(x+2y) \left( 1 + 2 \frac{dy}{dx} \right) + 1 \cdot \sin(x+2y) \]
\[ \frac{dy}{dx} = x \cos(x+2y) + 2x \cos(x+2y) \frac{dy}{dx} + \sin(x+2y) \]
\[ \therefore \frac{dy}{dx} \left[ 1 - 2x \cos(x+2y) \right] = x \cos(x+2y) + \sin(x+2y) \]
\[ \therefore \frac{dy}{dx} = \frac{x \cos(x+2y) + \sin(x+2y)}{1 - 2x \cos(x+2y)} \]
4. \[ y = f(x) = (3x + 4)^2 (3 - 2x)^3 \]
\[ f'(x) = (3x + 4)^2 \cdot 3 (3 - 2x)^3 (-2) + 2(3x + 4). 3. (3 - 2x)^3 \]
\[ = -6 (3x + 4)^2 (3 - 2x)^5 + 6 (3x + 4)(3 - 2x)^3 \]
\[ = 6 (3x + 4)(3 - 2x)^2 \left[ -(3x + 4) + (3 - 2x) \right] \]
\[ f'(x) = 6 (3x + 4)(3 - 2x)^2 (-5x - 1) \]
\[ \text{set} \]
\[ x = -\frac{4}{3}, \quad x = \frac{3}{2}, \quad \text{or} \quad x = -\frac{1}{5} \]

5. Given: \[ y = \frac{x}{\sqrt{4-x^2}} = \frac{x}{(4-x^2)^{\frac{1}{2}}} \]
\[ \frac{dy}{dx} = \frac{1}{(4-x^2)^{\frac{1}{2}}} \cdot \frac{(4-x^2)^{\frac{1}{2}} \cdot (4-x^2)^{\frac{1}{2}}}{(4-x^2)^{\frac{1}{2}}} \cdot \frac{(4-x^2)^{\frac{1}{2}}}{(4-x^2)^{\frac{1}{2}}} \]
\[ \frac{dy}{dx} = \frac{4-x^2 + x^2}{(4-x^2)^{\frac{3}{2}}} \]
\[ = 4 (4-x^2)^{-\frac{3}{2}} \]
\[ \frac{d^2y}{dx^2} = 4 \left( -\frac{3}{2} \right) (4-x^2)^{-\frac{5}{2}} \cdot (-1)x = \frac{12x}{(4-x^2)^{\frac{5}{2}}} \]
\[ \text{set} \]
\[ \frac{d^2y}{dx^2} = \frac{12x}{(4-x^2)^{\frac{5}{2}}} \]

6. Given: \[ f(x) = \sqrt{x^2 + 3} \]

NOTE that the point (3, 3) does not lie on the graph of \( y = \sqrt{x^2 + 3} \).
Thus, it is wrong to evaluate the derivative of \( y = \sqrt{x^2 + 3} \) at \( x = 3 \). This is why quite often it is a very good idea to draw a small sketch before doing the problem.

Let \( Q \) be the point of tangency. We can write \( Q(t, \sqrt{t^2 + 3}) \) for some \( t \in \mathbb{R} \).

Now, we can find the slope of \( PQ \) using 2 different methods:
\[ f'(x) = \frac{1}{2} (x^2 + 3)^{-\frac{1}{2}} \cdot x = \frac{x}{\sqrt{x^2 + 3}} \]

\[ \therefore \text{Slope of } PA = f'(t) = \frac{t}{\sqrt{t^2 + 3}} \]  

Also, slope of \( PQ = \frac{\sqrt{t^2 + 3} - 3}{t-3} \)  

Set 1 and 2 equal: \[ \therefore \frac{t}{\sqrt{t^2 + 3}} = \frac{\sqrt{t^2 + 3} - 3}{t-3} \]  

Cross multiply: \[ \frac{\sqrt{t^2 + 3} - 3}{t-3} \cdot \sqrt{t^2 + 3} = \sqrt{t^2 + 3} \cdot t \]

\[ \therefore \sqrt{t^2 + 3} = \sqrt{1+t} \]

Square both sides: \[ t^2 + 3 = 1 + 2t + t^2 \]

\[ t = 1. \]

\[ \therefore \text{The point } Q \text{ is } (t, \sqrt{t^2 + 3}) = (1, 2) = (x_1, y_1). \]

By 1, slope of \( PQ = m = \frac{1}{\sqrt{1^2 + 3}} = \frac{1}{2} \)

\[ \therefore \text{eqn. of PQ: } y - y_1 = m(x - x_1); \ y - 2 = \frac{1}{2}(x - 1) ; \ y = \frac{1}{2}x + \frac{3}{2}. \]

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7. Let \( r = \text{radius of sphere} \)

\( S = \text{Surface area} \)

\( V = \text{Volume of sphere} \)

Find: \[ \frac{dS}{dV} \bigg|_{r=2} \]

Since \( S = 4\pi r^2 \), \[ \frac{dS}{dr} = 8\pi r \]  

Since \( V = \frac{4}{3} \pi r^3 \), \[ \frac{dV}{dr} = 4\pi r^2 = \frac{dS}{dr} \]  

Now \[ \frac{dS}{dV} \bigg|_{r=2} = \frac{2}{r} \]

\[ \therefore \frac{dS}{dV} \bigg|_{r=2} = \frac{2}{r} \]  

un^2/\text{u}^3
Given \( f(x) = 6 \sin x + 2 \tan x - 5x \), where \( 0 \leq x \leq 2\pi \) with 
\[ x \neq \frac{\pi}{2} \text{ and } x \neq \frac{3\pi}{2} \]

(c) Domain = \( \left[ 0, \frac{\pi}{2} \right) \cup \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \cup \left( \frac{3\pi}{2}, 2\pi \right) \)

(d) Critical pts: 
\[ f'(x) = 6 \cos x + 2 \sec^2 x - 5 \]
\[ = 6 \cos x + \frac{2 \cos^2 x}{\cos^2 x} - 5 \]
\[ \therefore f'(x) = \frac{6 \cos^3 x - 5 \cos^2 x + 2}{\cos^2 x} \]

Zero

\( 6 \cos^3 x - 5 \cos^2 x + 2 = 0 \)

Let \( u = \cos x \)
\[ \therefore 6u^3 - 5u^2 + 2 = 0 \quad (1) \]

By inspection, or by using Rational Zero Theorem, one zero of (1) is \( u = -\frac{1}{2} \) [Because \( 6 \left( -\frac{1}{2} \right)^3 - 5 \left( -\frac{1}{2} \right)^2 + 2 = -\frac{3}{8} - \frac{5}{4} + 2 = 0 \)]
\[ \therefore u + \frac{1}{2} \text{ is a factor of } 6u^3 - 5u^2 + 2. \text{ To find other factors, } \]