MATH 200 (FALL 2003)  
TEST 2  
NAME:-------------------

**SHOW ALL WORK**

1. Find the derivatives of the following functions. Simplify your answers for parts (a) and (b).
   
   (a) \( y = \frac{3}{2\sqrt{x^2+1}} \)  
   (b) \( f(\theta) = \sin^2(3\theta) \)  
   (c) \( y = \sec^2 t + \sec(t^2) \)

2. Use the definition of the derivative to find \( f'(x) \) given that \( f(x) = -\frac{5}{4x^2} \). DO NOT use short-cut rules of differentiation.

3. Find the equation of the tangent line to the graph of \( f(x) = -2x^3 + 3x^2 - 4x + 5 \) at \( x = 2 \). Leave your answer in the slope-intercept form.

4. Find the critical points of the function \( y = (2x + 3)^4(2 - 3x)^2 \). Provide the exact answers.

5. Find the equation of the tangent line to the graph of \( x^2 + y^2 - 3xy + 2y = 1 \) at the point \( (1,1) \). Leave your answer in the slope-intercept form.

6. Find the points on the graph of \( f(x) = 54x + \frac{3}{2x^2} \) at which the tangent line is horizontal. Find both \( x \) and \( y \)-coordinates. Provide the exact answers.

7. A spherical balloon is inflated at a rate of \( 3 \text{ in}^3/\text{sec} \). Find the rate of change of the diameter of the balloon when the volume of the balloon is equal to \( 9\pi/2 \text{ in}^3 \). Provide the exact answer.

8. State the Mean Value Theorem (MVT) in precise terms. Draw a BIG diagram to illustrate the idea behind the theorem.

Consider the function \( f(x) = \frac{2x^2 - 1}{x + 1} \) on the closed interval \([0, 3]\). Check whether the MVT can be applied to this function. If so, find the \( c \)-values guaranteed by the theorem.

9. Find the absolute maximum/minimum of the function \( f(x) = 2x^4 - 4x^2 + 10 \) on the interval \([-2, 3]\).

   What theorem guarantees the existence of these absolute maximum and absolute minimum? State this theorem in precise terms.

10. A balloon rises vertically at a rate of \( 2 \text{ ft/sec} \) from a point on the ground 40 meters from an observer. Provide the exact answers to the following:

    (a) Find the rate of change of the angle of elevation of the balloon from the observer, when the balloon is 30 meters above the ground.
1. (a) \( y = \frac{3}{2\sqrt{x^2+1}} \) \:
   \( \frac{dy}{dx} = \frac{3}{2} \cdot \frac{1}{(x^2+1)^{\frac{3}{2}}} \cdot \frac{2x}{2(x^2+1)^{\frac{3}{2}}} = -\frac{3x}{2(x^2+1)^{\frac{3}{2}}} \)

(b) \( f(0) = \sin^2(3\theta) = [\sin(3\theta)]^2 \)
   \( f'(\theta) = 2[\sin(3\theta)] \cdot \cos(3\theta) \cdot 3 = 6\sin(3\theta)\cos(3\theta) \)

(c) \( y = \sec^2 t + \sec(t^2) = [\sec(t)]^2 + \sec(t^2) \)
   \( \frac{dy}{dt} = 2[\sec(t)] \cdot \sec(t)\tan(t) + \sec(t^2) \cdot \tan(t^2) \cdot 2t \)
   \( \therefore \frac{dy}{dt} = 2\sec^2 t \tan t + 2t \sec(t^2) \tan t^2 \)

2. Given \( f(x) = \frac{-5}{4x^2} \)
   \( f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{-5}{4(x+h)^2} - \frac{5}{4x^2} = \lim_{h \to 0} \frac{-5x^2 + 5(x+h)^2}{h \cdot 4x^2(x+h)^2} \)
   \( = \lim_{h \to 0} \frac{-5x^2 + 5x^2 + 10xh + 5h^2}{h \cdot 4x^2(x+h)^2} \)
   \( = \lim_{h \to 0} \frac{10x}{4x^2(x+h)^2} = \frac{10x}{4x^4} = \frac{5}{2x^3} \)
   \( \therefore f'(x) = \frac{5}{2x^3} \)

3. \( f(x) = -2x^3 + 3x^2 - 4x + 5 \) at \( x = 2 \)
   \( f(2) = -2(2)^3 + 3(2)^2 - 4(2) + 5 = -16 + 12 - 8 + 5 = -7 \)

Slope of tangent at \( x = 2 \) is \( f'(2) = (-m) \)
   \( f'(x) = -6x^2 + 6x - 4 \)
   \( f'(2) = -6(2)^2 + 6(2) - 4 = -24 + 12 - 4 = -16 = m \)

Equation of tangent line: \( y - y_1 = m(x - x_1) \)
   \( y + 7 = -16(x - 2) \)
   \( y = -16x + 25 \) is the required equation of the tangent line

4. Given: \( x^2 + y^2 - 3xy + 2y = 1 \) at \((1,1)\)

Differentiate both sides implicitly with respect to \( x \).

\[ 2x + 2y \cdot \frac{dy}{dx} - 3(x\frac{dy}{dx} + y) + 2\frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} (2y - 3x + 2) = -2x + 3y \]

\[ \frac{dy}{dx} = \frac{-2x + 3y}{2y - 3x + 2} \]

\[ m = \text{slope of tangent at } (1,1) = \frac{dy}{dx} \bigg|_{(1,1)} = \frac{-2(1)+3(1)}{2(1)-3(1)+2} = \frac{1}{1} = 1 \]

Equation of tangent line: \( y - y_1 = m(x - x_1) \)

\[ y - 1 = 1(x - 1) \]

\[ y = x \]
4) Given: \( x^2 + y^2 - 3xy + 2y = 1 \) at \((1,1)\) \(\therefore x_1 = 1\) and \(y_1 = 1\).

Differentiate both sides implicitly with respect to \(x\).

\[ 2x + 2y \cdot \frac{dy}{dx} - 3\left(x \frac{dy}{dx} + y\right) + 2 \frac{dy}{dx} = 0. \]

\[ \therefore \frac{dy}{dx} \left(2y - 3x + 2\right) = -2x + 3y \quad \therefore \frac{dy}{dx} = \frac{-2x + 3y}{2y - 3x + 2} \]

\[ m = \text{slope of tgt at } (1,1) = \frac{dy}{dx}\mid_{(1,1)} = \frac{-2(1) + 3(1)}{2(1) - 3(1) + 2} = \frac{1}{1} = 1 \]

\[ \therefore \text{eqn of tgt line: } y - y_1 = m(x - x_1) \quad : y - 1 = 1(x - 1) \quad : y = x \]

5) \( f(x) = 54x + \frac{3}{2x} = 54x + \frac{3}{2}x^{-1} \)

\[ \therefore f'(x) = 54 + \frac{3}{2}(-1)x^{-2} = 54 - \frac{3}{2x^2}. \]

Any horizontal tangent line has slope zero, so set \(f'(x) = 0\) and solve for \(x\).

\[ 54 - \frac{3}{2x^2} = 0 \quad : \frac{3}{2x^2} = 54 \quad : x^2 = \frac{3}{2(54)} = \frac{1}{36} \quad : x = \pm \sqrt{\frac{1}{36}} = \pm \frac{1}{6} \]

When \(x = \frac{1}{6}\) : \(f(\frac{1}{6}) = 54(\frac{1}{6}) + \frac{3}{2(\frac{1}{6})} = 9 + \frac{9}{2} = 18 \quad : \) one point is \((\frac{1}{6}, 18)\)

When \(x = -\frac{1}{6}\) : \(f(-\frac{1}{6}) = 54(-\frac{1}{6}) + \frac{3}{2(-\frac{1}{6})} = -18 \quad : \) the second pt is \((-\frac{1}{6}, -18)\).

\[ \therefore \text{the points on the graph at which the tangent lines are horizontal are } (\frac{1}{6}, 18) \text{ and } (-\frac{1}{6}, -18). \]
Let $V = \text{Vol}$; $r = \text{rad.}$ (of the balloon); $x = \text{diameter}$

Given: \( \frac{dv}{dt} = 3 \text{ in}^3/\text{sec.} \)

Find: \( \frac{dx}{dt} \bigg| \text{v} = \frac{9\pi}{2} \)

**Use Chain Rule:** \( \frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt} \)

But, Since \( V = \frac{4\pi r^3}{3} \), \( \frac{dv}{dr} = \frac{4\pi r^2}{} \)

\[ \therefore \frac{dv}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \quad \therefore \frac{dr}{dt} = \left( \frac{dv}{dt} \right) \cdot \frac{1}{4\pi r^2} = \frac{3}{4\pi r^2} \]

Find \( r \) when \( v = \frac{9\pi}{2} \):
\[ \frac{9\pi}{2} = \frac{4\pi r^3}{3} \quad \therefore r^3 = \frac{27}{8} \quad \therefore r = \sqrt[3]{\frac{27}{8}} = \frac{3}{2} \text{ in} \]

\[ \therefore \frac{dr}{dt} \bigg| \text{v} = \frac{9\pi}{2} = \frac{3}{4\pi} \left( \frac{3}{2} \right)^2 = \frac{1}{3\pi} \text{ in/sec.} \]

But, \( x = 2r \), so \( \frac{dx}{dt} = 2 \frac{dr}{dt} = 2 \left( \frac{1}{3\pi} \right) \quad \therefore \frac{dx}{dt} = \frac{2}{3\pi} \text{ in/sec}. \]

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7. **MVT Statement** (See page 170 of your text)

Given \( f(x) = \frac{2x^2 - 1}{x+1} \) on \([0,3]\). Since \( f \) is a rational function with domain \((-\infty, -1) \cup (-1, \infty)\), and since \( x = -1 \) does not belong to the given closed interval \([0,3]\), \( f \) is cts on \([0,3]\) and diff. on \((0,3)\). \( \therefore \) The Mean Value Theorem is applicable.

Let \( a = 0 \) and \( b = 3 \). Then \( \frac{f(b) - f(a)}{b-a} = \frac{f(3) - f(0)}{3-0} = \frac{\frac{17}{4} - (-1)}{3} = \frac{21}{12} = \frac{7}{4} \)

Also, \( f'(x) = \frac{(x+1) \cdot 4x - (2x^2 - 1)(1)}{(x+1)^2} = \frac{4x^2 + 4x - 2x^2 + 1}{(x+1)^2} = \frac{2x^2 + 4x + 1}{(x+1)^2} \)

Set \( f'(x) = \frac{f(b) - f(a)}{b-a} \), and solve for \( x \). \( \therefore \frac{2x^2 + 4x + 1}{(x+1)^2} = \frac{7}{4} \)

\[ \therefore 8x^2 + 16x + 4 = 7x^2 + 14x + 7 \quad \therefore x^2 + 2x - 3 = 0 \quad \therefore (x+3)(x-1) = 0 \]

\[ \therefore x = -3 \text{ or } x = 1 \quad \text{But } x = -3 \text{ does not belong to the open int. } (0,3) \]

\[ \therefore x = 1 \quad \therefore c = 1. \]
7. MVT Statement (See page 170 of your text)

Given \( f(x) = \frac{2x^2-1}{x+1} \) on \([0,3]\). Since \( f \) is a rational function with domain \((-\infty,-1) \cup (-1,\infty)\), and since \( x = -1 \) does not belong to the given closed interval \([0,3]\), \( f \) is cts on \([0,3]\) and diff. on \((0,3)\). \( \therefore \) The Mean Value Theorem is applicable.

Let \( a = 0 \) and \( b = 3 \). Then \( \frac{f(b) - f(a)}{b-a} = \frac{f(3) - f(0)}{3-0} = \frac{\frac{17}{4} - (-1)}{3} = \frac{21}{12} = \frac{7}{4} \).

Also, \( f'(x) = \frac{(x+1) \cdot 4x - (2x^2-1)(1)}{(x+1)^2} = \frac{4x^2 + 4x - 2x^2 + 1}{(x+1)^2} = \frac{2x^2 + 4x + 1}{(x+1)^2} \).

Set \( f'(x) = \frac{f(b) - f(a)}{b-a} \), and solve for \( x \).

\[ \frac{2x^2 + 4x + 1}{(x+1)^2} = \frac{7}{4} \]

\[ \therefore 8x^2 + 16x + 4 = 7x^2 + 14x + 7 \]

\[ 8x^2 + 2x - 3 = 0 \]

\[ \therefore \ x = -3 \text{ or } x = 1 \]

But \( x = -3 \) does not belong to the open int. \((0,3)\). \( \therefore \) \( x = 1 \).

Thus, \( c = 1 \).

8. \( f(x) = (1-2x)^x (4x-3) \) on \([0,1]\)

Crit. pts: \( f'(x) = (1-2x)^x (4x-3) - (1-2x)^x \cdot 2 \cdot (4x-3) = 4(1-2x)^x - 4(1-2x)(4x-3) \).

\[ f'(x) = 0 \implies x = \frac{1}{2} \text{ or } x = \frac{2}{3} \] (both pts are in the domain)

The only critical points are \( x = \frac{1}{2} \) and \( x = \frac{2}{3} \).

Now: \( f(0) = (1)^2 (-3) = -3 \) smallest

\( f\left(\frac{1}{2}\right) = 0 \)

\( f\left(\frac{2}{3}\right) = (1-\frac{4}{3})^\frac{2}{3} (-3) = \frac{1}{3} \cdot \frac{x}{3} = \frac{1}{27} \)

\( f(1) = (-1)^2 (1) = 1 \) largest

\( \therefore \) The abs. min. value of the function is \(-3\) (occurs at \( x = 0 \)) and the abs. max. value of the function is \(1\) (occurs at \( x = 1 \)).

The Extreme Value Thm guarantees the existence of the abs. max and abs. min for the above problem.

Extreme Value Theorem statement (See page 160 of your text).
Given \( \frac{dh}{dt} = 2 \text{ ft/sec} \)

(a) \( \frac{d\theta}{dt} \bigg|_{h=30} \)

First, \( \tan\theta = \frac{h}{40} \); \( h = 40 \tan\theta \).

\[ \therefore \frac{dh}{dt} = 40 \cdot \sec^2\theta \cdot \frac{d\theta}{dt} \]

\[ 2 = 40 \cdot \sec^2\theta \cdot \frac{d\theta}{dt} \]

\[ \therefore \frac{d\theta}{dt} = \frac{2}{40 \sec^2\theta} = \frac{\cos^2\theta}{20} \]

Now, find \( \theta \) when \( h=30 \):

\[ x^2 = h^2 + 40^2 \]

\[ \therefore x = \pm \sqrt{30^2 + 40^2} = 50 \text{ ft} \]

\[ \therefore \cos\theta = \frac{40}{x} = \frac{40}{50} = \frac{4}{5} \]

\[ \therefore \frac{d\theta}{dt} \bigg|_{h=30} = \frac{\cos^2\theta}{20} \bigg|_{\cos\theta = \frac{4}{5}} = \left(\frac{\frac{4}{5}}{20}\right) = \frac{4}{125} \text{ rad/Sec.} \]

(b) \( \frac{dx}{dt} \bigg|_{h=30} \)

First, \( \cos\theta = \frac{40}{x} \)

\[ \therefore \frac{dx}{dt} = 40 \sec\theta \cdot \tan\theta \cdot \frac{d\theta}{dt} \]

\[ \therefore \frac{dx}{dt} \bigg|_{h=30} = 40 \cdot \frac{1}{\cos \theta} \cdot \frac{(4)}{(4)} \cdot \frac{4}{125} = \frac{10}{125} \cdot \frac{4}{25} = \frac{30}{25} = \frac{6}{5} \text{ ft/sec.} \]