Given one trig function, finding the others

1. Given \( \sin \theta = \frac{-2}{\sqrt{5}} \) with \( \theta \) in quadrant \( III \), find the exact value of \( \cos \theta \).

There are two methods to do this. Learn both!

**Method I (Using \( x-y-r \) calculations)**

\[
\sin \theta = \frac{-2}{\sqrt{5}} = \frac{y}{r}
\]

::: we can pick \( y = -2 \), \( r = \sqrt{5} \)

(Important: This choice is legal as it agrees with the diagram. In quadrant \( III \) \( y \) is \( \Theta \), and \( r \) is always \( + \).)

Now \( y = -2 \); \( r = \sqrt{5} \); \( x = ? \)

Use \( r^2 = x^2 + y^2 \)

\[
(\sqrt{5})^2 = x^2 + (-2)^2
\]

\[
5 = x^2 + 4
\]

::: \( x^2 = 1 \)

::: \( x = \pm 1 \)

::: \( x = \pm \sqrt{1} \)

But \( \theta \) is in quadrant \( III \), so \( x \) must be \( \Theta \)

::: \( x = -1 \)

Finally, \( \cos \theta = \frac{x}{r} = \frac{-1}{\sqrt{5}} \)

::: \( \cos \theta = -\frac{1}{\sqrt{5}} \)
Method II (Use trigonometric identities)

Given: \( \sin \theta = -\frac{2}{\sqrt{5}} \) with \( \theta \) in \( \text{III} \)

Find: \( \cos \theta \)

Use \( \sin^2 \theta + \cos^2 \theta = 1 \)

\( \left( -\frac{2}{\sqrt{5}} \right)^2 + \cos^2 \theta = 1 \)

\( \frac{4}{5} + \cos^2 \theta = 1 \)

\( \cos^2 \theta = 1 - \frac{4}{5} \)

\( \cos^2 \theta = \frac{1}{5} \)

\( \cos \theta = \pm \sqrt{\frac{1}{5}} \)

But \( \theta \) is in quadrant \( \text{III} \), so \( \cos \theta \) must be negative.

\( \therefore \cos \theta = -\sqrt{\frac{1}{5}} \)

\( \therefore \cos \theta = -\frac{1}{\sqrt{5}} \)
Given \( \csc \beta = -\frac{\sqrt{7}}{2} \) with \( \beta \) in quadrant IV, find the exact value of \( \tan \beta \)

**Method I (Using \( x-y-r \) Calculations)**

\[
\csc \beta = -\frac{\sqrt{7}}{2} = \frac{r}{y}
\]

However, you cannot pick \( r = -\sqrt{7}, y = 2 \) as it does not agree with the diagram. This is because we know that \( r \) is always \( + \), and in quadrant IV, \( y \) is \( - \).

\[ \therefore \text{Write } \csc \beta = -\frac{\sqrt{7}}{2} = \frac{r}{y} \& \text{ then pick } r \text{ and } y \]

So, we can pick \( y = -2 ; \quad r = \sqrt{7} ; \quad x = ? \)

Use \( x^2 + y^2 = r^2 \)
\[
x^2 + (-2)^2 = (\sqrt{7})^2
\]
\[ \therefore x^2 + 4 = 7
\]
\[ \therefore x^2 = 3
\]
\[ \therefore x = \pm \sqrt{3}
\]

But \( x \) is in IV, so that \( x \) must be \( + \)

\[ \therefore x = +\sqrt{3}
\]

Finally, \( \tan \beta = \frac{y}{x} = -\frac{2}{\sqrt{3}} \)

\[ \therefore \tan \beta = -\frac{2}{\sqrt{3}} \]
Method II (Using Trigonometric Identities)

Given: \( \csc \beta = -\frac{\sqrt{7}}{2} \) with \( \beta \) in IV

Find: \( \tan \beta \)

There are several ways of doing this. One plan is to use \( \csc^2 \beta = 1 + \cot^2 \beta \) to find \( \cot \beta \) first. After finding \( \cot \beta \), use \( \tan \beta = \frac{1}{\cot \beta} \) to find \( \tan \beta \).

\[
\therefore \text{Start with: } \csc^2 \beta = 1 + \cot^2 \beta \\
\left(-\frac{\sqrt{7}}{2}\right)^2 = 1 + \cot^2 \beta \\
\frac{7}{4} = 1 + \cot^2 \beta \\
\therefore \cot^2 \beta = \frac{7}{4} - 1 \\
\cot^2 \beta = \frac{3}{4} \\
\therefore \cot \beta = \pm \sqrt{\frac{3}{4}} \\
\]

But \( \beta \) is in quadrant IV, so \( \cot \beta \) must be \( \theta \)

\[
\therefore \cot \beta = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2} \\
\]

Now use: \( \tan \beta = \frac{1}{\cot \beta} \)

\[
\therefore \tan \beta = \frac{1}{-\sqrt{\frac{3}{4}}} = -\frac{2}{\sqrt{3}} \\
\therefore \tan \beta = -\frac{2}{\sqrt{3}} \\
\]
Given $\sec \alpha = \frac{-4}{3}$ with $\alpha$ in quadrant II, find the exact value of $\cosec \alpha$.

**Method I (Using $x-y-r$ calculations)**

$\sec \alpha = \frac{-4}{3} = \frac{r}{x}$

But you **cannot** pick $r = -4$ and $x = 3$, as it does not agree with the diagram. This is because we know that $r$ is always positive and in II, $x$ is negative.

∴ Write $\sec \alpha = \frac{-4}{3} = \frac{r}{x}$ and then pick $r$ & $x$.

So we can pick: $x = -3$; $r = 4$; $y =$ ?

Use $x^2 + y^2 = r^2$

$$(-3)^2 + y^2 = (4)^2$$

$9 + y^2 = 16$

∴ $y^2 = 7$

∴ $y = \pm \sqrt{7}$

But $\alpha$ is in quadrant II, so $y$ must be positive.

∴ $y = +\sqrt{7}$

Finally, $\cosec \alpha = \frac{r}{y} = \frac{4}{\sqrt{7}}$

∴ $\cosec \alpha = \frac{4}{\sqrt{7}}$
Method III (Using trigonometric Identities)

Given: \( \sec \alpha = -\frac{4}{3} \) with \( \alpha \) in II

Find: \( \csc \alpha \)

There are several ways of doing this:

One plan is to use \( \sec^2 \alpha = 1 + \tan^2 \alpha \) to find \( \tan \alpha \). Then using values for \( \sec \alpha \) and \( \tan \alpha \), and by using the identity \( \frac{\sin \alpha}{\cos \alpha} = \tan \alpha \), you can find \( \csc \alpha \).

So: First use \( \sec^2 \alpha = 1 + \tan^2 \alpha \)

\( \left(-\frac{4}{3}\right)^2 = 1 + \tan^2 \alpha \)

\( \therefore \frac{16}{9} = 1 + \tan^2 \alpha \)

\( \therefore \tan^2 \alpha = \frac{16}{9} - 1 \)

\( \tan^2 \alpha = \frac{7}{9} \)

\( \therefore \tan \alpha = \pm \sqrt{\frac{7}{9}} \)

But \( \alpha \) is in quadrant II, so \( \tan \alpha \) must be \( \Theta \)

\( \therefore \tan \alpha = -\sqrt{\frac{7}{9}} = -\frac{\sqrt{7}}{3} \)

Now use \( \cos \alpha = \frac{1}{\sec \alpha} \) to find \( \cos \alpha \)

\( \therefore \cos \alpha = \frac{1}{-\frac{4}{3}} = -\frac{3}{4} \)
By (3) we know \( \tan \alpha \) and \( \cos \alpha \).

\[ \therefore \text{Use } \frac{\sin \alpha}{\cos \alpha} = \tan \alpha \text{ to find } \sin \alpha \]

\[ \therefore \sin \alpha = \cos \alpha \cdot \tan \alpha \] * USEFUL!

\[ \therefore \sin \alpha = \frac{-\sqrt{3}}{4} \times \frac{-\sqrt{7}}{4} = \frac{\sqrt{21}}{4} \]

\[ \therefore \sin \alpha = \frac{\sqrt{21}}{4} \]

Finally, use \( \csc \alpha = \frac{1}{\sin \alpha} \) to find \( \csc \alpha \)

\[ \therefore \csc \alpha = \frac{1}{\frac{\sqrt{21}}{4}} = \frac{4}{\sqrt{21}} \]

\[ \therefore \csc \alpha = \frac{4}{\sqrt{21}} \]