A BRIEF REVIEW OF ALGEBRA AND TRIGONOMETRY

Some Key Concepts:

1. The slope and the equation of a straight line
2. Functions and functional notation
3. The average rate of change of a function and the DIFFERENCE-QUOTIENT OF A FUNCTION
4. Domains of some common types of functions
5. Graphs of functions and piecewise functions
6. Expanding, factoring, and simplifying expressions
7. Solving algebraic equations
8. Basic trigonometry
9. Solving trigonometric equations
1. The slope and the equation of a straight line:

(a) Slope of a straight line

The slope of the line $PQ = m = \frac{\text{rise}}{\text{run}} = \frac{\text{change of } y}{\text{change of } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$

Four different kinds of lines and their slopes:

(b) Equation of a straight line

The equation of a straight line can be written in any one of the following three ways:

(i) The Point-Slope Form

The equation of a straight line that passes through the point $(x_1, y_1)$ and having slope $m$ is given by
(ii) **The Slope-Intercept Form**

The equation of a line with slope \( m \) and having the y-intercept \((0, b)\) is given by

\[ y = mx + b \]

(iii) **The Standard Form**

In this form, you move all \(x\) and \(y\) terms of the equation of the given line, to one side, and the constant term to the other side. It is normally written as

\[ Ax + By = C \]

**Example:**

1. Find the equation of the line that passes through the points \((-1, 2)\) and \((2, 9)\). Provide the exact answer in the standard form with integral coefficients.

Solution: Make sure to draw your own sketch (not included here). Let \( P(x_1, y_1) \) be \((-1, 2)\) and \( Q(x_2, y_2) \) be \((2, 9)\). Then the slope \( m \) of the line \( PQ \) is given by

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 2}{2 + 1} = \frac{7}{3} \]

The equation of this line \( PQ \) can be found either by using the point-slope form or by the slope-intercept form. For example, using the point-slope form \( y - y_1 = m(x - x_1) \), its equation is given by,

\[ y - 2 = \frac{7}{3}(x + 1) \]

Multiply both sides by 3 to get:

\[ 3(y - 2) = 7(x + 1) \]
\[ 3y - 6 = 7x + 7 \]

In the standard form, the equation is:

\[ 7x - 3y = -13 \]

(c) **Parallel and perpendicular lines:**

If two non-vertical lines are parallel, then they have equal slopes. In other words, if \( m_1 \) and \( m_2 \) are the slopes of the parallel lines, then \( m_1 = m_2 \). On the other hand, if two non-vertical lines are perpendicular, then the slope of one line is equal to the negative reciprocal of the slope of the other. In other words, if \( m_1 \) and \( m_2 \) are the slopes of the perpendicular lines, then \( m_1 = -1/m_2 \).
Example:  
1. Find the equation of the line that is perpendicular to the line \(2x - 3y = 4\) and which passes through the point \((-1,3)\). Provide the exact answer in the slope-intercept form.

Solution: Make sure to make a brief sketch – coordinate axes are not necessary (not included).

First, find the slope of the given line \(2x - 3y = 4\). To do this, arrange this equation in the form \(y = mx + b\) (i.e. solve for \(y\)) and look for the coefficient of \(x\):

\[
2x - 4 = 3y; \quad 3y = 2x - 4; \quad y = \frac{2}{3}x - \frac{4}{3}
\]

Therefore, the slope of the given line is \(2/3\).

So, the slope of a line perpendicular to the original line is the negative reciprocal of \(2/3\), i.e. \(-1/(2/3)\) or \(-3/2\).

Now, to find the equation of the required perpendicular line, use the point-slope form with \(m = -3/2\) and \((x_1, y_1) = (-1, 3)\):

\[
y - 3 = -\frac{3}{2}(x + 1)
\]

To put in the slope-intercept form, solve the above equation for \(y\):

\[
y = -\frac{3}{2}x + \frac{3}{2}
\]

2. Functions and the functional notation:

Quite often, a function can be given by an equation, such as \(y = 2x^2 - 3x + 4\). The same function can be re-written as \(f(x) = 2x^2 - 3x + 4\), which is called the functional notation. Thus, \(f(x)\) is just another name for “\(y\)”. Thus, as a specific example, \(f(3)\) means the \(y\)-value of the function when \(x\) is equal to 3.

So, this can be found by substituting each \(x\)-variable in the expression \(2x^2 - 3x + 4\) by “3”. Therefore, we have

\[
f(3) = 2(3)^2 - 3(3) + 4 = 18 - 9 + 4 = 13
\]

\[
\therefore \quad f(3) = 13
\]

Note that \(f(3) = 13\) just means that, when \(x = 3\), the \(y\)-value of the function \(f\) is equal to 13.
Examples:

1. Given \( f(x) = 2x + \sin(x) \), find the exact value of \( f(\pi/3) \).

Solution:

\[
\begin{align*}
 f\left(\frac{\pi}{3}\right) &= 2 \left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right) \\
&= \frac{2\pi}{3} + \frac{\sqrt{3}}{2} = \frac{4\pi + 3\sqrt{3}}{6} \\
\therefore \quad f\left(\frac{\pi}{3}\right) &= \frac{4\pi + 3\sqrt{3}}{6} 
\end{align*}
\]

2. Given \( f(x) = 3x^2 - 4x + 1 \), find \( x \) if \( f(x) = 21 \).

Solution:

DO NOT PLUG IN \( x = 21 \) in the given equation, as that would be a major mistake! Rather set \( f(x) = 21 \), and solve the resulting equation for \( x \), as follows:

\[
\begin{align*}
3x^2 - 4x + 1 &= 21 \\
3x^2 - 4x - 20 &= 0 \\
\text{The above is a quadratic equation. BE VERY FAMILIAR with the methods of solving quadratic equations and other equations. For example, one can use the Factoring Method or the Quadratic Formula } \\
&= \left(-b \pm \sqrt{b^2 - 4ac}\right)/(2a). \text{ Let us use the Factoring Method, since the right-hand side of the last equation can be found to be factorable:} \\
&= (3x - 10)(x + 2) = 0 \\
\text{Now use the zero-product property: } \quad \therefore \ 3x - 10 = 0 \text{ or } x + 2 = 0 \\
\therefore \quad x &= 10/3 \text{ or } x = -2 \\
\text{Thus, there are two } x \text{-values for which } f(x) = 21. 
\end{align*}
\]

3. VERY IMPORTANT EXAMPLE:

Given \( f(x) = 3x^2 - 4x + 1 \), find \( f(x + h) \). Expand and simplify your answer.

Solution:

This problem illustrates one of the subtle points of functional notation: DO NOT simply add “\( h \)” to the right-hand side of the equation to obtain \( 3x^2 - 4x + 1 + h \) as the answer for \( f(x + h) \). In fact, \( 3x^2 - 4x + 1 + h \) is equal to \( f(x) + h \), which is quite different from the desired \( f(x + h) \).

In order to find \( f(x + h) \), replace all the \( x \)-terms on the right hand-side of the given equation by \( x + h \):

\[
\therefore \quad f(x + h) = 3(x + h)^2 - 4(x + h) + 1
\]
Now use rules of algebra such as the Foil Method or special squaring formulas to expand the right-hand side:

\[ f(x + h) = 3(x^2 + 2xh + h^2) - 4x - 4h + 1 \]
\[ = 3x^2 + 6xh + 3h^2 - 4x - 4h + 1 \]
\[ \therefore f(x + h) = 3x^2 + 6xh + 3h^2 - 4x - 4h + 1 \]

Just to summarize, given a function \( f(x) \), the two quantities \( f(x + h) \) and \( f(x) + h \) are totally different!

For example, for the function \( f(x) = \sqrt{1-x} \), we can find these two quantities as follows:

\[ \text{Given: } f(x) = \sqrt{1-x} \]
\[ \therefore f(x + h) = \sqrt{1-(x+h)} = \sqrt{1-x-h} \]
\[ \text{But } f(x) + h = \sqrt{1-x} + h \]
\[ \text{In general, } f(x + h) \neq f(x) + h \]

### 3. Average rate of change of a function and the difference-quotient:

(a) The average rate of change of a function

In the above diagram, when the \( x \)-value is changed from \( x_1 \) to \( x_2 \), the \( y \)-value of the function \( f(x) \) changes from \( y_1 = f(x_1) \) to \( y_2 = f(x_2) \). Then, the change of \( x \), or the increment of \( x \), is given by \( x_2 - x_1 \), and is denoted by \( \Delta x \). Similarly, the change of \( y \), or the increment of \( y \), is given by \( y_2 - y_1 = f(x_2) - f(x_1) \), and is denoted by \( \Delta y \).
We define the average rate of change of \( f(x) \) from \( x = x_1 \) to \( x = x_2 \) as the ratio \( \Delta y / \Delta x \). It is very important to observe that, in the above diagram \( \Delta y / \Delta x \) is equal to the slope of the secant line \( PQ \). A secant line of a curve is a line joining any two points of the curve. Thus we have:

\[
\begin{align*}
\text{The average rate of change of } f(x) & \text{ from } x = x_1 \text{ to } x = x_2 \text{ is equal to any of } \\
\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \text{slope of the secant line } PQ
\end{align*}
\]

Examples:

1. Find the average rate of change of the function \( f(x) = \cos^2(2x) \) from \( x = 0 \) to \( x = \pi/3 \).

Solution

First make sure to draw your own diagram to see the corresponding secant line (not included). The graphical viewpoint is very essential for a successful study of especially calculus.

Let \( x_1 = 0 \) and \( x_2 = \pi/3 \). Then the average rate of change of \( f(x) \) from \( x = x_1 \) to \( x = x_2 \) is given by:

\[
\begin{align*}
\frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{f(\pi/3) - f(0)}{(\pi/3) - 0} = \frac{\cos^2(2\pi/3) - \cos^2(0)}{(\pi/3)} = \frac{(-1/2)^2 - 1^2}{(\pi/3)} = \frac{-3/4}{(\pi/3)} = -\frac{9}{4\pi}
\end{align*}
\]

(b) The difference-quotient of a function

In part (a), we just defined the average rate of change of a function \( f(x) \) from \( x = x_1 \) to \( x = x_2 \).

The difference quotient of a function is just a special case of the average rate of change – in other words, the difference-quotient is just the average rate of change of \( f(x) \) from \( x \) to \( x + h \). In order to find a formula for the difference-quotient, let \( x_1 = x \) and \( x_2 = x + h \), and rewrite the previous average rate of change of formula \( \frac{f(x_2) - f(x_1)}{x_2 - x_1} \) as follows:

\[
\text{Difference Quotient} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}
\]

Thus, one can look at the difference-quotient of a function in three different ways. As the algebraic meaning, or the definition, the difference-quotient of a function is \( \frac{f(x+h) - f(x)}{h} \). As the geometric meaning of the difference-quotient, it is equal to the slope of the secant line \( PQ \) where \( P(x, f(x)) \) and \( Q( x + h, f(x + h) ) \) are two points on the graph of \( y = f(x) \). As the physical meaning of the difference-quotient, it is equal to the average rate of change of the function \( f(x) \) from \( x \) to \( x + h \). All these ideas are illustrated in the writings below:
Difference-Quotient = \( \frac{f(x+h) - f(x)}{h} \)

**IDEA:**

\[ y = f(x) \]

\[ PQ \text{ is called a secant line} \]

\[ h \text{ is equal to the change of } x, \text{ and also denoted by } \Delta x \]

\[ f(x+h) - f(x) \text{ is equal to the change of } y, \text{ and also denoted by } \Delta y \]

\[ D. Q = \frac{f(x+h) - f(x)}{h} = \text{The slope of the secant line } PQ \]

\[ \text{Algebraic Meaning or definition} \]

\[ \text{Geometric Meaning} \]

\[ \text{Physical Meaning} \]

\[ \text{The average rate of change of } f \text{ from } x \text{ to } x+h \]
Here are some worked examples on difference-quotients of several types of functions:

**VERY IMPORTANT:** Given a function \( f(x) \), how to calculate the D. Q. of \( f \):

**Examples:**

1. Given \( f(x) = 2x^3 - x^2 + 5 \), find & completely simplify its diff-quotient.

\[
D.\ Q. = \frac{f(x+h) - f(x)}{h} = \frac{[2(x+h)^3 - (x+h)^2 + 5] - [2x^3 - x^2 + 5]}{h} = \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - (x^3 + 2xh + h^2) + 5 - 2x^3 + x^2 - 5}{h} = \frac{2x^3 + 6x^2h + 6xh^2 + 2h^3 - x^3 - 2xh - h^2 + 5 - 2x^3 + x^2 - 5}{h} = \frac{6x^3h + 6xh^2 + 2h^3 - 2xh - h^2}{h} = \frac{k(6x^2 + 6xh + 2h^2 - 2x - h)}{h} = 6x^2 + 6xh + 2h^2 - 2x - h
\]

**Note:** Assuming \( h \neq 0 \), you can cancel "h" term.

\[
\therefore D. Q. of f = \frac{f(x+h) - f(x)}{h} = 6x^2 + 6xh + 2h^2 - 2x - h
\]
(2) Given \( f(x) = \frac{-3}{1 - 2x} \), find & completely simplify its difference-quotient

\[
D. \ a = \frac{f(x+h) - f(x)}{h}
\]

\[
= \left[ \frac{-3}{1 - 2(x+h)} \right] - \left[ \frac{-3}{1 - 2x} \right]
\]

\[
= \frac{-3}{(1 - 2x - 2h)} + \frac{3}{(1 - 2x)}
\]

\[
= \left[ \frac{-3}{(1 - 2x - 2h)} + \frac{3}{(1 - 2x)} \right] \cdot \frac{(1 - 2x - 2h)(1 - 2x)}{h \cdot (1 - 2x - 2h)(1 - 2x)}
\]

\[
= \frac{-3}{(1 - 2x - 2h)} \cdot \frac{(1 - 2x - 2h)(1 - 2x)}{h \cdot (1 - 2x - 2h)(1 - 2x)} + \frac{3}{(1 - 2x)} \cdot \frac{(1 - 2x - 2h)(1 - 2x)}{h \cdot (1 - 2x - 2h)(1 - 2x)}
\]

\[
= \frac{-3(1 - 2x) + 3(1 - 2x - 2h)}{h \cdot (1 - 2x - 2h)(1 - 2x)}
\]

\[
= \frac{-6 + 6x + 6 - 6x - 6h}{h \cdot (1 - 2x - 2h)(1 - 2x)}
\]

\[
= \frac{-6}{h(1 - 2x - 2h)(1 - 2x)} \]

\[\text{Note: Assuming } h \neq 0, \text{ you can cancel the } h \text{'s}\]

\[
\therefore \ D. \ a = \frac{f(x+h) - f(x)}{h} = \frac{-6}{(1 - 2x - 2h)(1 - 2x)}
\]
③ Given \( f(x) = \sqrt{1-2x} \) do the same.

D. \( Q = \frac{f(x+h) - f(x)}{h} \)

\[ = \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} \]

Now rationalize the numerator by multiplying the num & den by the conjugate of \( \sqrt{1-2x-2h} - \sqrt{1-2x} \)

\[ = \frac{(\sqrt{1-2x-2h} - \sqrt{1-2x}) \cdot (\sqrt{1-2x-2h} + \sqrt{1-2x})}{h \cdot (\sqrt{1-2x-2h} + \sqrt{1-2x})} \]

\[ = \frac{(1-2x-2h) - (1-2x)}{h \cdot (\sqrt{1-2x-2h} + \sqrt{1-2x})} \]

\[ = \frac{1-2x-2h-1+2x}{h \cdot (\sqrt{1-2x-2h} + \sqrt{1-2x})} \]

\[ = \frac{-2h}{h \cdot (\sqrt{1-2x-2h} + \sqrt{1-2x})} \]

Note: Assuming \( h \neq 0 \), you can cancel "h" terms

\[ \therefore D. Q = \frac{f(x+h)-f(x)}{h} = \frac{-2}{(\sqrt{1-2x-2h} + \sqrt{1-2x})} \]
**Exercise:** Given the following functions, find and completely simplify the Diff-Quot.
(Simplify so that the denominator "h" term cancels out)

1. \( f(x) = -3x^2 + 4x - 5 \)
2. \( f(x) = 2x - 3x^3 \)
3. \( f(x) = \frac{5}{3x-4} \)
4. \( f(x) = \frac{-6}{\sqrt{3-4x}} \)
5. \( f(x) = 2x - \frac{3}{x} \)
6. \( f(x) = \frac{1+2x}{2-3x} \)
7. \( f(x) = \frac{2}{(1-4x)^2} \)
4. Domains of some common types of functions

Given a function \( f(x) \), its domain is the set of all values of \( x \), for which \( f(x) \) is defined. As a very simple example, the domain of the function \( f(x) = x^2 \) is the entire real line, or the interval \((-\infty, \infty)\), since the expression \( x^2 \) is defined for any real value. The function \( f(x) = x^2 \) is just one example of a polynomial. In fact, the domain of any polynomial is the entire real line.

However, on the other hand, the domain of the function \( f(x) = \frac{1}{x-2} \) is not the entire real line. Note that the expression \( \frac{1}{x-2} \) is not defined at \( x = 2 \), since \( 1/0 \) is not defined. However, for any other \( x \)-value, the expression \( \frac{1}{x-2} \) is always defined. In other words, the domain of the function \( f(x) = \frac{1}{x-2} \) is all real numbers except \( x = 2 \). In the interval notation, we can write this domain as \((-\infty, 2) \cup (2, \infty)\).

Here are two very useful principles:

\[
\text{For } \frac{1}{T} \text{ to be defined, } T \text{ must be a nonzero quantity, i.e. } T \neq 0
\]

\[
\text{For } \sqrt{T} \text{ to be defined as a real number, } T \text{ must be a nonnegative quantity, i.e. } T \geq 0
\]

Examples:

1. Find the domain of \( f(x) = \frac{x}{2x^2 - 16} \). Express the exact answer in the interval notation.

Solution:

The numerator “\( x \)’’ is always defined. By the previous remarks, the domain of \( f(x) \) is all \( x \)-values such that the denominator \( 2x^2 - 16 \) is nonzero. In order to find when this does happen, ask the opposite question, i.e. set \( 2x^2 - 16 \) equal to zero, and solve for \( x \).

\[
2x^2 - 16 = 0
\]
\[
2x^2 = 16
\]
\[
x^2 = 8
\]
\[
\therefore \quad x = \pm 2\sqrt{2}
\]

However, DO NOT be confused in saying that domain is equal to \( \pm 2\sqrt{2} \). The domain is in fact, all real values except \( \pm 2\sqrt{2} \). Thus, in the interval notation, the domain of \( f(x) \) is equal to \((-\infty, -2\sqrt{2}) \cup (-2\sqrt{2}, 2\sqrt{2}) \cup (2\sqrt{2}, \infty)\).
2. Find the domain of \( f(x) = \sqrt{3 - 7x} \). Express the exact answer in the interval notation.

Solution:
Recall that \( \sqrt{T} \) to be defined as a real number, \( T \) must be a nonnegative quantity, i.e. \( T \geq 0 \).
Therefore, the domain of \( f(x) \) is all \( x \)-values such that \(- 3x + 7 \geq 0\). We can solve this linear inequality as follows:

\[
-3x + 7 \geq 0 \\
-3x \geq -7
\]
Now divide both sides of the inequality by \(-3\). However, note that when you multiply or divide both sides of an equality by an negative quantity, the inequality sign must be reversed!!

\[
\frac{-3x}{-3} \leq \frac{-7}{-3} \\
\therefore x \leq \frac{7}{3}
\]
Therefore, the domain of \( f(x) \) is all real values less than or equal to \( 7/3 \). In the interval notation, the answer is \( (-\infty, 7/3 ] \).

5. Graphs of functions and piecewise functions
(a) Some basic graphs
Your study of calculus will become more enjoyable, if you can memorize the shapes of some important graphs:

(a) \( y = x \)  
(b) \( y = x^2 \)  
(c) \( y = x^3 \)  
(d) \( y = |x| \)  
(e) \( y = \sqrt{x} \)  
(f) \( y = \frac{1}{\sqrt{x}} \)
Now here is the main advantage: Once you memorize the above shapes, you can build the graphs of more complicated functions using transformation techniques.

**Examples:**

1. Draw a quick graph of \( f(x) = |x + 2| - 3 \).

Solution:

Note that the parent function for the given equation is \( y = |x| \), and we already know the shape of its graph by part (d) above. Now start with this graph (d), translate it 2 units to the *left*, and then 3 units *down*, to obtain the required graph. The vertex of the final graph is \((-2, -3)\).
(b) Piecewise functions and their graphs

One of the biggest concepts of calculus is the notion of the limit of a function. In order to understand this concept successfully, it is better to be familiar with the idea of piecewise functions. These functions normally have two or more pieces in its defining equation. Here is an example:

**Examples:**

1. Consider the function given by $f(x) = \begin{cases} 
  x + 2 & \text{if } x < 2 \\
  x^2 + 2 & \text{if } 2 \leq x < 4 \\
  18 & \text{if } x > 4 
\end{cases}$

(a) Find $f(-3)$   (b) Find $f(3)$   (c) Find $f(4)$   (d) Draw a clear graph of $f(x)$

**Solution:**

(a) Note that $x = -3$ satisfies the inequality $x < 2$. Therefore, to find $f(-3)$, plug in $x = -3$ in the expression $x + 2$, the first “piece” of the piecewise function. Thus, $f(-3) = -3 + 2 = -1$.

(b) Here $x = 3$ satisfies the inequality $2 \leq x < 4$. Therefore, to find $f(3)$, we use the second “piece” of the given function. Thus, $f(3) = (3)^2 + 2 = 11$.

(c) This part is VERY INTERSTING! In this case, $x = 4$ does not satisfy either inequality $2 \leq x < 4$ or $x > 4$. Therefore, $f(4)$ is undefined, or simply does not exist. What this means is that, the graph of $f(x)$ will have a hole corresponding to $x = 4$. See the graph in part (d) below.

(d) First the graph the three functions $y = x + 2$, $y = x^2 + 2$, and $y = 18$ by completing disregarding the conditions given by the inequalities. This is where knowing the basic graphs will really pay dividends!
Now we are ready to cut each of the above graphs using appropriate vertical lines: As shown above, cut the first graph using the vertical line $x = 2$, and take the part to the left of this vertical line. The second graph must be cut using two vertical lines, $x = 2$ and $x = 4$, and take the part of the graph between these two vertical lines. The third graph is cut by the vertical line $x = 4$, and take the part to the right of this vertical line.

Finally, we must glue together the remaining parts of these three graphs. However, one must take care of the end points of the pieces which are glued together. The final assembled graph is given below – note that a solid dot means the corresponding point is included, while a small circle means the point is excluded.
6. Expanding, factoring, and simplifying expressions

Here are some crucial algebra formulas:

(A) Expansion Formulas:

1. \((a + b)^2 = a^2 + 2ab + b^2\)
2. \((a - b)^2 = a^2 - 2ab + b^2\)
3. \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)
4. \((a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\)

(B) Factoring Formulas:

1. \(a^2 + b^2 \) does not have any real factors. However, the complex factors are \((a-bi)(a+bi)\)
2. \(a^2 - b^2 = (a-b)(a+b)\)
3. \(a^3 + b^3 = (a+b)(a^2 - ab + b^2)\)
4. \(a^3 - b^3 = (a-b)(a^2 + ab + b^2)\)

Examples:

1. Factor: \(16x^3 - 54\)
   Solution:
   
   \(16x^3 - 54 = 2(8x^3 - 27) = 2[(2x)^3 - (3)^3] = 2(2x-3)(4x^2 + 6x + 9)\)

2. Factor: \(x^4 - 16y^4\)
   Solution:
   
   \(x^4 - 16y^4 = (x^2)^2 - (4y^2)^2 = (x^2 - 4y^2)(x^2 + 4y^2) = (x-2y)(x+2y)(x^2 + 4y^2)\)

3. Factor: \(x^3 - x^2y - xy^2 + y^3\)
   Solution:
   Use factoring by grouping method.
   
   \(x^3 - x^2y - xy^2 + y^3 = x^2(x - y) - y^2(x - y) = (x - y)(x^2 - y^2) = (x - y)(x - y)(x + y) = (x - y)^2(x + y)\)
4. **VERY IMPORTANT (LEARN THIS!!)**

Simplify the following algebraic expression, and leave the answer in the factored form:

\[(x^2 + 4)^2\cdot 4(2x + 3)^3(2) + 2(x^2 + 4)\cdot 2(x + 3)^4\]

Solution:

\[(x^2 + 4)^2\cdot 4(2x + 3)^3(2) + 2(x^2 + 4)\cdot 2(x + 3)^4 = 8(x^2 + 4)^2(2x + 3)^3 + 4x(x^2 + 4)(2x + 3)^4\]

\[= 4(x^2 + 4)(2x + 3)^3 [2(x^2 + 4) + x(2x + 3)]\]

\[= 4(x^2 + 4)(2x + 3)^3 [2x^2 + 8 + 2x^2 + 3x]\]

\[= 4(x^2 + 4)(2x + 3)^3 (4x^2 + 3x + 8)\]

5. **VERY IMPORTANT (LEARN THIS!!)**

Simplify the following expression, and leave the answer in the factored form:  \[2 - \frac{9}{2x^2}\]

Solution:

\[2 - \frac{9}{2x^2} = \frac{2}{1} - \frac{9}{2x^2} = \frac{4x^2 - 9}{2x^2} = \frac{(2x - 3)(2x + 3)}{2x^2}\]

6. **EXTREMELY IMPORTANT (LEARN THIS - YOU WILL NEED THIS SEVERAL TIMES!!)**

Simplify the following expression completely: \[x \cdot \frac{1}{2}(x^2 - 4)^{-1/2} \cdot (2x) + (x^2 - 4)^{1/2}\]

Solution:

\[x \cdot \frac{1}{2}(x^2 - 4)^{-1/2} \cdot (2x) + (x^2 - 4)^{1/2} = \frac{x^2}{(x^2 - 4)^{1/2}} + (x^2 - 4)^{1/2}\]

\[= \frac{x^2}{(x^2 - 4)^{1/2}} + \frac{(x^2 - 4)^{1/2}}{1}\]

\[= \frac{x^2 + (x^2 - 4)}{(x^2 - 4)^{1/2}}\]

\[= \frac{2x^2 - 4}{\sqrt{x^2 - 4}} = \frac{2(x^2 - 2)}{\sqrt{x^2 - 4}}\]
Simplify the following expression completely:

\[
\frac{(9 - x^2)^{1/2} \cdot 2x - x^2 \cdot \frac{1}{2}(9 - x^2)^{-1/2} (-2x)}{(9 - x^2)}
\]

Solution:

\[
\frac{2x(9 - x^2)^{1/2}}{(9 - x^2)} + \frac{x^3}{(9 - x^2)^{1/2}}
\]

\[
\frac{[2x(9 - x^2)^{1/2} + \frac{x^3}{(9 - x^2)^{1/2}}] 
\cdot (9 - x^2)^{1/2}}{(9 - x^2)^2}
\]

\[
\frac{2x(9 - x^2) + x^3}{(9 - x^2)^{3/2}}
\]

\[
\frac{x [18 - 2x^2 + x^2]}{(9 - x^2)^{3/2}} = \frac{x(18 - x^2)}{(9 - x^2)^{3/2}}
\]

**REMINDER:** Now try several problems from the transparency on “Some Factoring Problems”

**7. Solving algebraic equations**

Solving various types of equations is an essential skill for anyone studying calculus. Here are some examples.

**Examples:**

1. Solve \(x^4 + 4x^2 + 2 = 0\)

Solution:

The above is not a quadratic equation. However, with a simple substitution, it can be transformed into a quadratic equation. Let \(u = x^2\). Then the given equation becomes:

\[u^2 + 4u + 2 = 0\]

The right-hand side of the above equation does not factor. So use the quadratic formula with \(a = 1, b = 4, \text{ and } c = 2\).
\[
\therefore \quad u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(2)}}{2(1)} = \frac{-4 \pm \sqrt{16}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}
\]

\[
x^2 = -2 \pm \sqrt{2}
\]

\[
\therefore \quad x = \pm \sqrt{-2 \pm \sqrt{2}}
\]

Therefore, the four solutions are \(x = -\sqrt{2} - \sqrt{2}, -\sqrt{2} + \sqrt{2}, \sqrt{2} - \sqrt{2}, \) and \(\sqrt{2} + \sqrt{2}\)

2. Solve \( (4x + 3)^2 (2x - 1)^3 + (4x + 3)^3 (2x - 1)^2 = 0 \)

**Solution:**

DO NOT expand the right-hand side of the given equation – instead factor the right-hand side

\[
\therefore \quad (4x + 3)^2 (2x - 1)^2 [(2x - 1) + (4x + 3)] = 0
\]

\[
(4x + 3)^2 (2x - 1)^2 (6x + 2) = 0
\]

\[
2(4x + 3)^2 (2x - 1)^2 (3x + 1) = 0
\]

Use the zero product property to get: \(4x + 3 = 0\) or \(2x - 1 = 0\) or \(3x + 1 = 0\)

\[
x = -3/4, 1/2, -1/3
\]

**REMINDER:** Now try several problems from the transparency on “Some Equation Solving Problems”
8. Basic trigonometry

Make every attempt to be familiar with the following facts from trigonometry:

(a) Trig functions of special acute angles

(b) Trig functions of all special angles in “color wheel”

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \theta )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{2}{\sqrt{3}} )</td>
<td>( \frac{3}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \cos \theta )</td>
<td>1</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( \tan \theta )</td>
<td>0</td>
<td>( \frac{\sqrt{3}}{3} )</td>
<td>1</td>
<td>( \sqrt{3} )</td>
<td>undefined</td>
</tr>
</tbody>
</table>

(c) Signs of trigonometric functions in various quadrants
(d) Basic Trigonometric Identities

(e) The Double Angle Formulas:

Examples:

1. Find the exact value of the expression \(2\cos^3\left(\frac{2\pi}{3}\right) + 3\tan^2\left(\frac{\pi}{6}\right) - 4\cos^3(3\pi)\)

Solution:

\[
2\cos^3\left(\frac{2\pi}{3}\right) + 3\tan^2\left(\frac{\pi}{6}\right) - 4\cos^3(3\pi) \\
= 2\left(-\frac{1}{2}\right)^3 + 3\left(\frac{1}{\sqrt{3}}\right)^2 - 4(-1)^3 = 2\left(-\frac{1}{8}\right) + 3\left(\frac{1}{3}\right) - 4(-1) = \frac{19}{4}
\]

2. **SOMETHING VERY USEFUL TO KNOW!**

Find all the values for which Sine, Cosine, and Tangent functions are zero.

Solution:

Just use the information gained by above (a) and (b).

(a) \(\sin\theta\) is zero for any angle \(\theta\) that falls on the \(x\)-axis. In other words, this happens for \(\theta = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots\). All these values are integer multiples of \(\pi\). Thus we can conclude the following:

\(\sin\theta = 0 \text{ if and only if } \theta = n\pi\) where \(n\) is any integer
(b) \( \cos \theta = 0 \) is zero for any angle \( \theta \) that falls on the \( y \)-axis. In other words, this happens for \( \theta = \ldots, -3\pi/2, -\pi/2, \pi/2, 3\pi/2, 5\pi/2, \ldots \). All these values are odd integer multiples of \( \pi/2 \). Thus we can conclude the following:

\[
\cos \theta = 0 \quad \text{if and only if} \quad \theta = n\frac{\pi}{2} \quad \text{where} \quad n \text{ is any odd integer.}
\]

(c) Just like \( \sin \theta \), \( \tan \theta \) is zero for any angle \( \theta \) that falls on the \( x \)-axis. This we can write:

\[
\tan \theta = 0 \quad \text{if and only if} \quad \theta = n\pi \quad \text{where} \quad n \text{ is any integer}
\]

9. Solving trigonometric equations

Solving trigonometric equations quite often uses the facts (a) and (b) of the previous section 8.

Examples:

Try your BEST to understand the first two examples. The others are built from these two.

1. Solve \( 2\cos \theta + 1 = 0 \) where \( 0 \leq \theta < 4\pi \).

Solution:

First, isolate the trigonometric function by solving for \( \cos \theta \):

\[
2\cos \theta = -1
\]

\[
\cos \theta = -\frac{1}{2}
\]

Now draw the quadrant diagram, and use the “ASTC” principle to find out which quadrant the angle \( \theta \) belongs to. We know that \( \cos \theta \) is a negative number. So the angle \( \theta \) must belong to either quadrant II or quadrant III. We will locate these two quadrants by circling them as given in the diagram below:

The next step is to find the reference angle. It is the acute angle between the \( X \)-axis and the terminal side of the angle \( \theta \). So, by dropping the negative side of the right-hand-side of the equation \( \cos \theta = -1/2 \), we ask that the cosine of what acute angle is equal to +1/2. It is 60° or \( \pi/3 \) radians. Thus, the reference angle is equal to \( \pi/3 \). We will now draw this reference angle in the circled quadrants:
Thus, there are 4 solutions to the given equation. They are \( \theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \) and \( \frac{10\pi}{3} \).

2. Solve \( 4\cos^2\theta - 3 = 0 \) where \( 0 \leq \theta < 2\pi \).

Solution:

\[
\therefore 4\cos^2\theta = 3
\]

\[
\cos^2\theta = \frac{3}{4}
\]

\[
\cos\theta = \pm \frac{\sqrt{3}}{2}
\]

We can treat the above like two separate equations, \( \cos\theta = -\frac{\sqrt{3}}{2} \) and \( \cos\theta = \frac{\sqrt{3}}{2} \) and solve them. For both of these equations, the reference angle is 30° or \( \pi/6 \) radians.

(i) To solve \( \cos\theta = -\frac{\sqrt{3}}{2} \), observe that \( \cos\theta \) is negative only in quadrants II and III. Draw the reference angle of \( \pi/6 \) radians in these two quadrants as follows:

Therefore, the two solutions for this part are \( 5\pi/6 \) and \( 7\pi/6 \).
(ii) On the other hand, to solve \( \cos \theta = \frac{\sqrt{3}}{2} \), observe that \( \cos \theta \) is positive in quadrants I and IV. Draw the reference angle of \( \pi/6 \) radians in these two quadrants as follows:

![Reference Angle Diagram]

Therefore, the two solutions for this part are \( \pi/6 \) and \( 11\pi/6 \).

The final solutions are obtained by combining the answers for the above parts (i) and (ii). Therefore, the solutions are \( \theta = \pi/6, 5\pi/6, 7\pi/6, \) and \( 11\pi/6 \).

3. Solve \( \sin(2\theta) + 4\sin \theta = 0 \) where \( -2\pi < \theta < 4\pi \)

Solution:
Use the Double Angle Formula \( \sin(2\theta) = 2\sin \theta \cos \theta \) to convert the entire right-hand side of the equation to the single angle \( \theta \), as follows:

\[
2\sin \theta \cos \theta + 4\sin \theta = 0 \\
\therefore 2\sin \theta (\cos \theta + 2) = 0 \\
\sin \theta = 0 \text{ or } \cos \theta + 2 = 0 \\
\sin \theta = 0 \text{ or } \cos \theta = -2
\]

Now solve like two different problems. Make sure to supply your own diagrams (omitted here).

For any \( \theta \), \( \cos \theta \) is always between -1 and 1, so the equation \( \cos \theta = -2 \) doesn’t have any solutions. Thus, only solutions for the original equation can come from \( \sin \theta = 0 \). However, in example 2 of section 8 we learned that \( \sin \theta = 0 \) only happens when the angle \( \theta \) lies along the x-axis. Therefore, all the solutions strictly between \( -2\pi \) and \( 4\pi \) are \( -\pi, 0, \pi, 2\pi, \) and \( 3\pi \).
4. Solve $\cos(2x) - \sin(x) = -2$ where $0 \leq x < 2\pi$.

Solution:
Use the Double Angle Formula $\cos(2x) = 1 - 2\sin^2 x$ to convert the entire right-hand side of the equation to the single angle $x$. Also, in the process, the entire equation will be in terms of one type of trigonometric function, i.e. in terms of $\sin(x)$ only.

\[
(1 - 2\sin^2(x)) - \sin(x) = -2 \\
2\sin^2(x) + \sin(x) - 3 = 0 \\
(2\sin x + 3)(\sin x - 1) = 0 \\
\therefore \sin(x) = -3/2 \quad \text{or} \quad \sin(x) = 1
\]

Now solve like two different problems. Make sure to supply your own diagrams (omitted here).

First, the equation $\sin(x) = -3/2$ does not have any solutions, since for any angle $x$, $\sin(x)$ always lies between -1 and 1. On the other hand, the only solution for $\sin(x) = 1$ is $x = \pi/2$ in the given domain $0 \leq x < 2\pi$. Therefore, the only solution to the given equation is $x = \pi/2$. ■