The Polar and Iwasawa Decompositions
of $SO_o(n, 1), n \geq 2$

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Abstract

The two decompositions in the title of a semisimple Lie group are well-establsihed, see [2]. In this paper, we prove these decompositions of $G = SO_o(n, 1)$ by elementary methods using the geometry of $G/K$ where $K \simeq SO(n)$.

1 The Hyperboloid and a Cartan Involution

Consider the $(n+1)$-by-$(n+1)$ matrix

$$L = \begin{pmatrix} -I_n & 0_{n,1} \\ 0_{n,1}^T & 1 \end{pmatrix}$$

where $I_n$ is the identity $n$-by-$n$ matrix, $0_{n,1}$ is the $n$-by-1 zero matrix, and $T$ denotes the transpose of a matrix. For any two real vectors $x = (x_1, ..., x_{n+1})$ and $y = (y_1, ..., y_{n+1})$ we associate a real number according to the rule

$$x \cdot y = xI_{n,1}y^T = -\sum_{i=1}^{n} x_i y_i + x_{n+1}y_{n+1}. \quad (1.1)$$

The set

$$\mathcal{H} = \{ x \in \mathbb{R}^{n+1} : x \cdot x = 1, x_{n+1} > 0 \}.$$ 

is the hyperboloid model for $n$-dimensional hyperbolic geometry.

Let $SO_o(n, 1)$ be the connected component of

$$SO(n, 1) = \{ A \in SL(n+1, \mathbb{R}) : A^T LA = L \}$$

that contains the identity. In [3], it is shown that

$$G \equiv SO_o(n, 1) = \{ A \in SO(n, 1) : NA^T \in \mathcal{H} \} \quad (1.2)$$
where $N = (0, ..., 0, 1)$. Also, $G$ acts transitively on the hyperboloid $\mathbb{H}$ by the mapping

$$(A, x) \in G \times \mathbb{H} \mapsto xA^T \in \mathbb{H}.$$ 

The stabilizer of $N$ is the subgroup

$$K = \{ g \in G : gN = N \} = \left\{ \begin{pmatrix} A & 0_{n,1} \\ 0_{n,1}^T & 1 \end{pmatrix} : A \in SO(n) \right\}.$$ 

Since the action of $G$ is continuous, the mapping $\phi : G/K \to \mathbb{H}$ given by

$$\phi(gK) = Ng^T \quad (1.3)$$

is a diffeomorphism, see [2, page 124].

The Lie algebras $\mathcal{G}$ and $\mathcal{K}$ of $G$ and $K$, respectively, are

$$\mathcal{G} = \left\{ \begin{pmatrix} X_n & X_{n,1} \\ X_{n,1}^T & 0 \end{pmatrix} : X_n \text{ is skew symmetric of order } n, \ X_{n,1} \text{ is an } n \times 1 \text{ matrix} \right\}, \quad \text{and}$$

$$\mathcal{K} = \left\{ \begin{pmatrix} X_n & 0_{n,1} \\ 0_{n,1}^T & 0 \end{pmatrix} : X_n \text{ is skew symmetric of order } n \right\}.$$ 

Consider the involution

$$\theta : \mathcal{G} \to \mathcal{G} \quad (1.4)$$
given by

$$\theta(A) = LAL = -A^T.$$ 

The eigenspaces of $\theta$ are

$$\{ A \in \mathcal{G} : \theta(A) = A \} = \mathcal{K}, \quad \text{and}$$

$$\{ A \in \mathcal{G} : \theta(A) = -A \} \equiv \mathcal{P} = \left\{ \begin{pmatrix} 0_n & X_{n,1} \\ X_{n,1}^T & 0 \end{pmatrix} : X_{n,1} \text{ is an } n \times 1 \text{ matrix} \right\}$$

where $0_n$ is the zero $n \times n$ matrix. In addition,

$$\mathcal{G} = \mathcal{K} + \mathcal{P} \quad (1.5)$$

is a vector space direct sum. A straightforward verification shows that the bilinear form on $\mathcal{G} \times \mathcal{G}$ defined

$$\langle A, B \rangle = -Tr(A\theta(B)), \quad A, B \in \mathcal{G} \quad (1.6)$$

is positive-definite. In fact,

$$\langle A, A \rangle = \sum_{i,j=1}^{n+1} A_{i,j}^2, \quad \text{if } A = (A_{i,j}) \in \mathcal{G}.$$ 

The involution in (1.4) is an example of a Cartan involution [2, page 185].
2 Geometry of $G/K$ and $IH$

We denote the exponential of a matrix by $e^A$ or $\exp(A)$ where $A \in G$. If $W$ is a set of matrices, we write

$$\exp(W) = \{e^A : A \in W\}.$$  

Let $\Pi : G \to G/K$ be the map given by $\Pi(g) = gK$. If the identity coset is denoted $\circ = eK$, then the differential of $\Pi$ at the identity in $G$ is the linear map

$$d\Pi : G \to T_\circ G/K \quad (2.1)$$

given by

$$d\Pi(x) = \frac{d}{dt} \bigg|_{t=0} [e^{tx}K], \ x \in G.$$  

**Lemma 1** Suppose $\mathcal{M}$ is a subspace of $\mathcal{G}$ such that

$$\mathcal{G} = \mathcal{K} + \mathcal{M}$$

is a vector space direct sum. Then the restriction $d\Pi \big|_{\mathcal{M}}$ is a linear isomorphism from $\mathcal{M}$ onto $T_\circ G/K$.

**Proof** Choose open subsets $N_1$ and $N_2$ such that

$$0 \in N_1 \subseteq \mathcal{M}, \ 0 \in N_2 \subseteq \mathcal{K}$$

and where the mapping

$$(x, y) \mapsto \exp(x)\exp(y) \quad (2.2)$$

is a diffeomorphism from $N_1 \times N_2$ onto an open neighborhood of the identity in $G$, see Lemma 4 in [2, page 115].

By taking a smaller $N_2$, if necessary, we may assume that the exponential mapping is a diffeomorphism from $N_2$ onto an open neighborhood of the identity in $K$. Since $K$ has the relative topology of $G$, let $V$ be an open neighborhood in $G$ containing the identity such that

$$\exp(N_2) = V \cap K.$$  

Choose a compact neighborhood $U$ of $0$ in $\mathcal{M}$ such that

$$U \subseteq N_1$$

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and
\[ \exp(U)\exp(-U) \subseteq V. \] (2.3)

If
\[ \Pi(\exp(u_1)) = \Pi(\exp(u_2)), \quad u_1, u_2 \in U \]

then
\[ \exp(-u_2)\exp(u_1) \in K \cap V = \exp(N_2). \]

Thus, \( u_1 = u_2 \) due to the bijection in (2.2). In particular, the restriction of \( \Pi \) to \( \exp(U) \) is injective.

Let \( U^\circ \) be the interior of \( U \) in \( \mathcal{M} \). Since \( \exp(U^\circ)\exp(N_2) \) is open in \( G \) and \( \Pi \) is an open mapping, the set
\[ \Pi(\exp(U^\circ)\exp(N_2)) = \Pi(\exp(U^\circ)) \]

is open in \( G/K \). A discussion in [2, page 123] shows that the mapping
\[ \psi : x \in U^\circ \mapsto e^xK \] (2.4)

is a chart on the manifold \( G/K \). Then the differential \( d\psi \) of \( \psi \) is a linear isomorphism between the tangent spaces \( T_xU^\circ \) and \( T_xG/K \). Since \( T_xU^\circ = \mathcal{M} \), we have
\[ d\psi = d\Pi\bigg|_{\mathcal{M}}. \]

This proves the lemma. \( \square \)

To each \( g \in G \), let \( \tau(g) \) be the diffeomorphism of \( G/K \) defined by left multiplication
\[ \tau(g)(xK) = gxK, \quad x \in G. \]

The differential of \( \tau(g) \) at \( \circ \) is a linear isomorphism from \( T_oG/K \) onto \( T_{gK}(G/K) \) where
\[ d\tau(g)(d\Pi x) = \left. \frac{d}{dt} \right|_{t=0} \left[ ge^{tx}K \right], \quad x \in \mathcal{M}. \]

If \( \mathcal{M} = \mathcal{P} \) in Lemma 1, then the bilinear form \( Q_{gK} \) on the tangent space \( T_{gK}(G/K) \) given by
\[ Q_{gK}\left( d\tau(g)d\Pi(x_1), d\tau(g)d\Pi(x_2) \right) = \langle x_1, x_2 \rangle, \quad x_1, x_2 \in \mathcal{P}. \] (2.5)

is well-defined, see the discussion in [2, page 210].
Next, we describe the tangent spaces in $\mathbb{H}$.

**Lemma 2** The tangent space at $x \in \mathbb{H}$, namely,

$$T_x \mathbb{H} = \{ v \in \mathbb{R}^{n+1} : v \star x = 0 \}$$

(2.6)

is an $n$-dimensional real vector space. Moreover, the bilinear form on $T_x \mathbb{H}$ given by

$$(v, w)_x \equiv -(v \star w), \quad v, w \in T_x \mathbb{H}$$

(2.7)

is positive-definite.

**Proof** We omit the proof and refer the reader to [1] and [3].

Since the mapping $\phi$ in (1.3) is a diffeomorphism from $G/K$ onto the hyperboloid $\mathbb{H}$, the differential $d\phi$ is a linear isomorphism. In fact,

$$d\phi : T_gK(G/K) \rightarrow T_{Ng} \mathbb{H}$$

satisfies

$$d\phi(d\tau(g)d\Pi x) = \left. \frac{d}{dt} \right|_{t=0} \phi(ge^{tx}K) = \left. \frac{d}{dt} \right|_{t=0} N(ge^{tx})^T = N(gx)^T$$

(2.8)

for $x \in \mathcal{M}$.

**Lemma 3** The mapping $\phi : G/K \rightarrow \mathbb{H}$ in (1.3) is an isometry of Riemannian manifolds.

**Proof** It suffices to verify

$$Q_{gK} \left( d\tau(g)d\Pi(x_1), d\tau(g)d\Pi(x_2) \right) = \left( d\phi(d\tau(g)d\Pi x_1), d\phi(d\tau(g)d\Pi x_2) \right)_{Ng}$$

for $x_1, x_2 \in \mathcal{P}$. However, this is a straightforward calculation using (2.5) and (2.8).

In [3], the geodesics emanating from $N$ are given by

$$\gamma_x(t) = \begin{cases} \cosh(t\|x\|_N)N + \frac{\sinh(t\|x\|_N)}{\|x\|_N}x & \text{if } x \neq 0 \\ N & \text{if } x = 0 \end{cases}$$

(2.9)

where $x \in T_N \mathbb{H}$ and $\|x\|_N = \langle x, x \rangle_N$.  

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Using the geodesics, we may define the exponential mapping on $T_N\mathcal{H}$.

**Definition 1**

The exponential map $\text{Exp}_{\mathcal{H}}$ from $T_N\mathcal{H}$ into $\mathcal{H}$ is given by

$$\text{Exp}_{\mathcal{H}}(x) = \gamma_x(1), \ x \in T_N\mathcal{H}. \quad (2.10)$$

If we pull-back the exponential mapping $\text{Exp}_{\mathcal{H}}$ of $\mathcal{H}$ using $\phi$, then we obtain the exponential map

$$\text{Exp} : T_\phi G/K \rightarrow G/K$$

for $G/K$. Namely, $\text{Exp}$ satisfies

$$\text{Exp}(d\Pi x) = \phi^{-1}\left(\text{Exp}_{\mathcal{H}}\left(d\phi d\Pi(x)\right)\right), \ x \in \mathcal{M} \quad (2.11)$$

However, if $x \in \mathcal{P}$ then

$$\text{Exp}_{\mathcal{H}}(d\phi d\Pi x) = \text{Exp}_{\mathcal{H}}(N x^T)$$

$$= \cosh(\|N x\|_N) N + \frac{\sinh(\|N x\|_N)}{\|N x\|_N} N x \quad \text{by (2.8)}$$

Moreover, a careful calculation shows

$$N e^x = \cosh(\|N x\|_N) N + \frac{\sinh(\|N x\|_N)}{\|N x\|_N} N x \quad \text{if } x \in \mathcal{P}.$$  

Thus, we have

$$\text{Exp}_{\mathcal{H}}(d\phi d\Pi x) = N(e^x)^T = \phi(e^x K), \ x \in \mathcal{P}.$$  

Consequently, the exponential map in (2.11) satisfies

$$\text{Exp}(d\Pi x) = e^x K, \ x \in \mathcal{P}$$

which is consistent with [2, page 212].
Corollary 4 Polar Decomposition
The mapping from $\mathcal{P}$ onto $G/K$ defined by

$$x \mapsto e^x K$$

(2.12)

is a diffeomorphism. Consequently, if $g \in G$ there exists a unique pair $(x, k) \in \mathcal{P} \times K$ such that

$$g = e^x k.$$

Proof In [3], it is proved that $\Exp$ is a surjective mapping. Since $\phi$ is bijective, it follows from identity (2.11) that $\Exp$ is a surjective mapping from $T_0 G/K$ onto $G/K$.

To show that the mapping in (2.12) is injective, we note that the matrices in $\mathcal{P}$ are real symmetric matrices. Then $e^x$ is a positive-definite real symmetric matrix for $x \in \mathcal{P}$. Since a non-singular real matrix can be written uniquely as a product of a positive-definite real symmetric matrix and a real orthogonal matrix, it follows that the mapping in (2.12) is injective.

3 Iwasawa Decomposition at the Lie Algebra Level

Let $e_{ij}$ be a square matrix of order $(n+1)$ whose only nonzero entry is its $(i, j)$-entry and which is 1. Using the identity

$$e_{ij}e_{kl} = \delta_{jk}e_{il}$$

we find

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}. \quad (3.1)$$

For $1 \leq i < j \leq n$ and $1 \leq k \leq n$, let

$$W_{i,j} = e_{ij} - e_{ji}, \quad V_k = e_{k,n+1} + e_{n+1,k}.$$

Consider the following eigenspaces of the endomorphism $ad(V_1)$ of $\mathcal{G}$:

$$\mathcal{G}_0 = \{X \in [V_1, X] = 0_{n+1}\}, \quad \mathcal{G}_1 = \{X \in \mathcal{G} : [V_1, X] = X\}, \quad \text{and} \quad \mathcal{G}_{-1} = \{X \in \mathcal{G} : [V_1, X] = -X\}.$$

Let $LS\mathcal{W}$ denote the linear space spanned by the elements of a set $\mathcal{W}$.

Corollary 5 Root Space Decomposition

1. $\mathcal{G}_0 = LS\{V_1, W_{i,j} : 2 \leq i < j \leq n\}$
2. \( G_i = \text{LS}\{V_j + W_{1,j} : 2 \leq j \leq n\} \)

3. \( G_{-1} = \text{LS}\{-V_j + W_{1,j} : 2 \leq j \leq n\} \)

4. \( G = G_0 + G_1 + G_{-1} \) is a direct sum of vector subspaces.

**Proof** For \( 2 \leq i < j \leq n \), we have

\[
[V_1, W_{i,j}] = 0_{n+1}, \quad [V_1, V_j + W_{1,j}] = V_j + W_{1,j}, \quad \text{and} \quad [V_1, -V_j + W_{1,j}] = -(-V_j + W_{1,j}).
\]

From which we obtain

\[
\begin{align*}
\text{LS}\{V_1, W_{i,j} : 2 \leq i < j \leq n\} & \subseteq G_0 & (3.2) \\
\text{LS}\{V_j + W_{1,j} : 2 \leq j \leq n\} & \subseteq G_1 & (3.3) \\
\text{LS}\{-V_j + W_{1,j} : 2 \leq j \leq n\} & \subseteq G_{-1}. & (3.4)
\end{align*}
\]

Since the following sets

\[
B_K = \{W_{i,j} : 1 \leq i < j \leq n\} \quad \text{and} \quad B_P = \{V_k : 1 \leq k \leq n\}
\]

are bases for the subspaces \( K \) and \( P \), respectively, and

\[ G = K + P \]

is a direct sum, it follows that equality holds in (3.2), (3.3), and (3.4). Consequently, the vector sum in part 4) is a direct sum. \( \square \)

Note, the 1-dimensional subspace

\[ H_P = \{aV_1 : a \in \mathbb{R}\} \]

is a maximal abelian subspace of \( P \) since

\[ [V_k, V_s] = W_{k,s}, \quad k < s. \]

**Corollary 6** *Iwasawa Decomposition of \( G \)*

\( G = K + H_P + G_1 \) is a direct sum of Lie algebras.

**Proof** Using Corollary 5, we obtain that

\[ G = K + H_P + G_1 \]
is a direct sum of vector subspaces. Since $\mathcal{K}$ and $\mathcal{H}_P$ are Lie subalgebras, it suffices to show that $\mathcal{G}_1$ is a Lie subalgebra.

Let $V_j + W_{1,j}, V_k + W_{1,k} \in \mathcal{G}_1$ for $2 \leq j, k \leq n$. Using Jacobi’s identity, we find

$$[V_1, [V_j + W_{1,j}, V_k + W_{1,k}]] = -[V_j + W_{1,j}, [V_k + W_{1,k}, V_1]] - [V_k + W_{1,k}, [V_1, V_j + W_{1,j}]]$$

$$= [V_j + W_{1,j}, V_k + W_{1,k}] - [V_k + W_{1,k}, V_j + W_{1,j}]$$

$$= 2[V_j + W_{1,j}, V_k + W_{1,k}].$$

However, by Corollary 5, the only eigenvalues of $ad(V_1)$ are 0 and ±1. Thus, we conclude

$$[V_j + W_{1,j}, V_k + W_{1,k}] = 0_{n+1}.$$

Hence, $\mathcal{G}_1$ is an abelian subalgebra.

4 Iwasawa Decomposition at the Group Level

Let $A$, $N$, and $S$ be the subgroups of $S_0(n, 1)$ generated by $\exp(\mathcal{H}_P)$, $\exp(\mathcal{G}_1)$, and $\exp(\mathcal{H}_P + \mathcal{G}_1)$, respectively. The proof in Theorem 2.1, [2, page 113] shows that $A$, $N$, and $S$ are connected Lie subgroups of $G$.

Since $\mathcal{H}_P$ and $\mathcal{G}_1$ are abelian Lie subalgebras, we have

$$A = \exp(\mathcal{H}_P) \quad \text{and} \quad N = \exp(\mathcal{G}_1). \quad (4.1)$$

**Lemma 7** $\exp$ is a diffeomorphism from $\mathcal{H}_P$ onto $A$.

**Proof** If $t \in \mathbb{R}$, then

$$e^{tV_1} = \begin{pmatrix} \cosh t & 0_{1,n-1} & \sinh t \\ 0_{n-1,1} & I_{n-1} & 0_{n-1,1} \\ \sinh t & 0_{1,n-1} & \cosh t \end{pmatrix}. \quad (4.2)$$

Thus, the exponential map on $\mathcal{H}_P$ is injective. \qed

**Lemma 8** $\exp$ is a diffeomorphism from $\mathcal{G}_1$ onto $N$.

**Proof** It suffices to show that the exponential mapping is injective on $\mathcal{G}_1$. Let $t_j \in \mathbb{R}$ where $2 \leq j \leq n$. Note,

$$\left(t_j(V_j + W_{1,j})\right)^3 = 0_{n+1}.$$
Then the exponential of the matrix \( t_j(V_j + W_{1,j}) \) is

\[
\exp\left( t_j(V_j + W_{1,j}) \right) = \begin{pmatrix}
1 - \frac{1}{2}t_j^2 & \cdots & 1 \\
-\frac{1}{2}t_j & \cdots & 1 \\
-1 & \cdots & 1 \\
\end{pmatrix}^{t_j} 
\]

(4.3)

where the \( j \)th row and \( j \)th column are enclosed by the vertical and horizontal lines.

Since \( G_1 \) is abelian, we have

\[
\exp\left( \sum_{j=2}^{n} t_j(V_j + W_{1,j}) \right) = \prod_{j=2}^{n} \exp\left( t_j(V_j + W_{1,j}) \right). 
\]

(4.4)

By induction, we find that the matrix product (4.4) is

\[
\begin{pmatrix}
1 - \frac{1}{2} \sum_{j=2}^{n} t_j^2 & \cdots & 1 \\
-\frac{1}{2} t_2 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
-\frac{1}{2} t_n & \cdots & 1 \\
\end{pmatrix}
\]

(4.5)

Thus, the exponential mapping on \( G_1 \) is injective.

\[ \square \]

**Lemma 9** \( S = AN \)

**Proof** If \( x \in H_\mathcal{P} \) and \( y \in G_1 \), then

\[
e^{y}e^{x} = e^{x}e^{-x}e^{y}e^{x} = e^{x}e^{(e^{-x}y) e^{x}}
\]

\[
= e^{x}e^{(e^{-ad_{x}})(y))}
\]

\[
= e^{x}e^{y/n}.
\]

Thus,

\[
AN = \{e^{x}e^{y} : (x, y) \in H_\mathcal{P} + G_1 \}
\]

(4.6)

is a subgroup of \( G \). Using the matrix representations in (4.2) and (4.5), we find the first column of the matrix product

\[
\exp(tV_1)\exp\left( \sum_{j=2}^{n} t_j(V_j + W_{1,j}) \right)
\]

is a subgroup of \( G \). Using the matrix representations in (4.2) and (4.5), we find the first column of the matrix product
is of the form

\[(\bullet, -t_2, \ldots, -t_n, \bullet)^T\]

where $T$ denotes the transpose of a matrix and the bullets $\bullet$’s denote the first and last entries of the first row. Consequently, the mapping

\[(x, y) \in \mathcal{H}_P \times \mathcal{G}_1 \mapsto e^x e^y \in AN \quad (4.7)\]

is bijective. Then $AN$ is a closed subgroup of $G$.

Since $A$ and $N$ are connected and group multiplication is a continuous function, it follows that $AN$ is connected. Then $AN$ is a connected Lie subgroup of $G$. The dimension of $AN$ is the same as the dimension of $S$ since the mapping in (4.7) is a diffeomorphism. Since there is a one-to-one correspondence between the Lie subalgebras of $\mathcal{G}$ and the connected Lie subgroups of $G$, it follows that $S = AN$. This completes the proof of the lemma.

\[\square\]

**Theorem 10** Iwasawa Decomposition of $G$

The mapping

\[(K_1, A_1, N_1) \mapsto K_1 A_1 N_1 \quad (4.8)\]

is a diffeomorphism from $K \times A \times N$ onto $G$.

**Proof** Denote the $S$-orbit in $G/K$ by

\[S_* = \{sK : s \in S\} \].

We will show that $S_*$ is open and closed in $G/K$. Since $G/K$ is connected, these imply that $S_* = G/K$. If $g \in G$, there exists $s \in S$ such that

\[g^{-1}K = s^{-1}K.\]

Thus, $g^{-1} = s^{-1}k^{-1}$ for some $k \in K$. By Lemma 9, we obtain $g = ks = kan$ for some $a \in A$, $n \in N$.

We will now show that $S_*$ is open. In the proof of Lemma 1 where

\[\mathcal{M} = \mathcal{H}_P + \mathcal{G}_1\]

we described a chart $\psi$ on the manifold $G/K$, see (2.4). Namely, there exists an open subset $U^\circ$ of 0 in $\mathcal{H}_P + \mathcal{G}_1$ such that the mapping

\[\psi : x \in U^\circ \mapsto e^x K\]
is a diffeomorphism onto an open subset of $G/K$. In particular,

$$\{e^xK : x \in U^o\}$$

is an open neighborhood of the identity coset $\circ$ in $G/K$. Then $S_\circ$ contains a neighborhood of $\circ$ in $G/K$. Using translations, we conclude that $S_\circ$ is open in $G/K$.

To show that $S_\circ$ is closed in $G/K$, suppose

$$\lim_{n \to \infty} \Pi(s_n) = xK, \ s_n \in S.$$ 

Since $\Pi(\exp(U^o))$ is an open neighborhood of $\circ$,

$$x^{-1}s_n = \exp(u_n)k_n, \ k_n \in K, u_n \in U^o$$

for large $n$. Using the compactness of $U$ and $K$ and by passing to subsequences if necessary, we may assume $\{u_n\}$ and $\{k_n\}$ are convergent sequences with limits $u_0 \in U$ and $k_0 \in K$, respectively. Then

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} x\exp(u_n)k_n = x\exp(u_0)k_0$$

and

$$xK = \lim_{n \to \infty} (s_nK) = x\exp(u_0)K.$$ 

Applying (2.3), we obtain

$$\exp(u_0) \in V \cap K = \exp(N_2).$$ 

Thus, by (2.2), we obtain $u_0 = 0$. Since $S$ is a closed subgroup of $G$, we find

$$\lim_{n \to \infty} s_n = xk_0 \in S$$ 

and $xK \in \Pi(S)$. Thus, $\Pi(S)$ is closed in $G/K$. Hence, the mapping (4.8) in the theorem is surjective.

Finally, we have to prove that (4.8) is injective. Suppose we have

$$K_1A_1N_1 = K_2A_2N_2$$

(4.10)

where $(K_1, A_1, N_1), (K_2, A_2, N_2) \in K \times A \times N$. Since $AN$ is a group, we may rearrange (4.10) and write

$$K_3 = A_3N_3$$

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where $K_3 = K_2^{-1}K_1$ and $A_3N_3 = (A_2N_2)(A_1N_1)^{-1}$. Using the matrix representations (4.2) and (4.5), we have

$$A_3N_3 = \begin{pmatrix}
\bullet & t_2 & \cdots & t_n & \bullet \\
-t_2 & 1 & & t_2 & \\
\vdots & & \ddots & \vdots & \\
-t_n & & & 1 & t_n \\
\bullet & t_2 & \cdots & t_n & \bullet
\end{pmatrix} \in K. \tag{4.11}$$

Then $t_2 = \cdots = t_n = 0$ and $N_3 = I_{n+1}$. Thus, $K_3 = A_3$ and consequently $K_3 = A_3 = I_{n+1}$.

This completes the proof of the theorem.

\[ \square \]

**Corollary 11** *Iwasawa Decomposition of G*

For each $X \in SO(n,1)$, there exists a unique $(A, t, t_2, \cdots, t_n) \in SO(n) \times \mathbb{R}^n$ such that

$$X = \begin{pmatrix}
A & 0_{n,1} \\
0_{n,1}^T & 1
\end{pmatrix}
\begin{pmatrix}
\cosh t & 0_{1,n-1} & \sinh t \\
0_{n-1,1} & I_{n-1} & 0_{n-1,1} \\
\sinh t & 0_{1,n-1} & \cosh t
\end{pmatrix}
\begin{pmatrix}
1 - \frac{1}{2} \sum_{j=2}^{n} t_j^2 & t_2 & \cdots & t_n \\
t_2 & 1 & & \vdots \\
\vdots & & \ddots & \vdots \\
t_n & & & 1 \\
-\frac{1}{2} \sum_{j=2}^{n} t_j^2 & t_2 & \cdots & t_n & 1 + \frac{1}{2} \sum_{j=2}^{n} t_j^2
\end{pmatrix}$$

**References**

