Skew Product Actions of Semi-Direct Product Groups

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Introduction

Let $f$ be a homomorphism from a second countable locally compact group $G$ into the continuous automorphisms of a second countable compact group $K$. Let $g * k = f(g)(k)$ and suppose further the mapping given by $(g, k) \rightarrow g * k$ is continuous from $G \times K$ into $K$, $g \in G, k \in K$. Then the cartesian product $K \times G$ becomes a group with multiplication given by $(k_1, g_1)(k_2, g_2) = (k_1 g_1 k_2, g_1 g_2)$, where $(k_i, g_i) \in K \times G, i = 1, 2$. We denote this semi-direct product group by $K \rtimes_s G$.

Let $(S, \mu)$, a standard Borel $K \times_s G$-space with a probability invariant measure $\mu$. In this paper we state necessary and sufficient conditions so the quotient Borel $K \times_s G$-mapping from $S$ into the space of $K$-orbits in $S$ has relative discrete spectrum. These conditions are in terms of the stabilizers and the natural action of $G$ on the dual $\hat{K}$ of $K$. (We remind the reader $\hat{K}$ is the set of unitary equivalence classes of irreducible unitary representations of $K$).

Choose a Borel subset $\overline{S}$ of $S$ that meets each $K$-orbit in $S$ exactly once. Let $p : (S, \mu) \rightarrow (\overline{S}, \overline{\mu})$ be a mapping given by $p(s) = \overline{s}$ where $\overline{s}$ is the unique point in $\overline{S}$ that meets the $K$-orbit $s \cdot K$ of $s$ and $\overline{\mu} = \mu \circ p^{-1}$.

$K \times_s G$ induces a natural action on the space of $K$-orbits in $S$. Since $\overline{S}$ is identified canonically with the space of $K$-orbits then $K \times_s G$ acts on $\overline{S}$. We will denote this new action by $\circ$. Namely, $K$ acts trivially on $\overline{S}$ and $G$ acts on $\overline{S}$ by $\overline{\pi} \circ g \equiv p(\overline{s} \cdot g)$ where $\cdot$ is the original action of $K \times_s G$ on $S$. Then $(S, \mu)$ becomes an extension of $(\overline{S}, \overline{\mu})$. (see [2] for definitions).

We will assume the stabilizers in $K$ of each $\overline{s}$ in $\overline{S}$ are all the same. For each $\overline{s} \in \overline{S}$, let $L = \{k \in K : \overline{s} \cdot k = \overline{s}\}$ and let $L^\perp$ be the set of all representations $\gamma$ in $\hat{K}$ whose restriction to $L$ contains the identity one dimensional representation as a direct summand. Let $G$ act on $\hat{K}$ by $(\pi \cdot g)(k) = \pi(g \cdot k)$ where $(k, g) \in K \times_s G, \pi \in \hat{K}$. The main result of this paper is the following.

**Theorem:** The extension $p : S \rightarrow \overline{S}$ has relatively discrete spectrum if and only if $\gamma \cdot G$ is a finite subset of $\hat{K}$ for each $\gamma \in L^\perp$.

To simplify the proof of the Theorem we first have a short discussion on the cocycle representation associated to the extension $p$. Let $\mu = \int \mu_\pi \, d\overline{\mu}(\overline{s})$ be a disintegration of $\mu$ over the fibers of $p$ where $\mu_\pi$ is a $K$-invariant measure on $S$ concentrated on $\overline{s} \cdot K$, and if $\mu_\pi \cdot g$ is the measure given by $(\mu_\pi \cdot g)(E) \equiv \mu_\pi(E \cdot g^{-1})$, $E$ a Borel subset of $S$, then $\mu_{\overline{s} \cdot g} = \mu_{\overline{s}} \cdot g$, for each $(\overline{s}, g) \in \overline{S} \times G$. 

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The natural cocycle representation $R$ associated to $p$ acts on the Hilbert bundle $\{L^2(S, \mu_\pi) : \pi \in \overline{S}\}$ over $\overline{S}$. For each $(k, g) \in K \times_s G$ and $\pi \in \overline{S}$, $R(\pi, (k, g)) : L^2(S, \mu_{\pi,g}) \to L^2(S, \mu_\pi)$ is a unitary operator given by (see [2], page 382):

$$R(\pi, (k, g)) h(s_1) = h(s_1 \cdot (k, g))$$

where $s_1 \in S, h \in L^2(S, \mu_{\pi,g})$.

Consider the bijection between $S$ and $\{(\pi, Lk) : \pi \in \overline{S}, k \in K\}$ given by $s \mapsto (\pi, Lk)$ where $\pi \cdot k = s$. If $(k_1, g) \in K \times_s G$ then the corresponding action of $K \times_s G$ on $\{(\pi, Lk) : \pi \in \overline{S}, k \in K\}$ is given by

1. $(\pi, Lk) \cdot k_1 = (\pi, Lkk_1)$

and

2. $(\pi, Lk) \cdot g = (\pi \circ g, Lb(\pi, g)(g^{-1} \cdot k))$ for some $b(\pi, g) \in K$ where $\pi \circ g \cdot b(\pi, g) = \pi \cdot g$.

Let $\psi_\pi : L\backslash K \to \pi \cdot K$ be the mapping given by $\psi_\pi(Lk_1) = \pi \cdot k_1$. If $m$ is the unique $K$-invariant probability measure on $L\backslash K$ then there is a non-negative real-valued Borel function $\phi$ defined on $\overline{S}$ satisfying $\mu_\pi = \phi(\pi) \cdot (m \circ \psi_\pi^{-1})$ for each $\pi \in \overline{S}$. Since $G$ acts ergodically on $(\overline{S}, \pi)$ and $\phi$ is $G$-invariant, then $\phi(\pi) = 1$ for $\pi$-a.e. $\pi$.

For each $\pi \in \overline{S}$ define a linear mapping

$$(\psi_\pi)_* : L^2(S, \mu_\pi) \to L^2(L\backslash K, m)$$

by $(\psi_\pi)_*(h) = h \circ \psi_\pi, h \in L^2(S, \mu_\pi)$. $(\psi_\pi)_*$ is a unitary operator if $\phi(\pi) = 1$. Hence, for each $(k, g) \in K \times_s G$ we can realize $R(\pi, (k, g)) : L^2(L\backslash K, m) \to L^2(L\backslash K, m)$ as a unitary operator for $\pi$-a.e. $\pi$, given by

$$R(\pi, (k, g)) h(Lk_1) = h(Lb(\pi, g)(g^{-1} \cdot (k_1 k)))$$

where $h \in L^2(L\backslash K, m), Lk_1 \in L\backslash K$. If $e$ is the identity element then $b(\pi, e) \in L$ for any $\pi \in \overline{S}$. Thus, $R(\pi, k)$ is independent of $\pi$ for any $k \in K$. Let $\pi$ be a unitary representation of $K$ on $L^2(L\backslash K, m)$ defined by $\pi(k) = R(\pi, k), \pi \in \overline{S}, k \in K$.

**Proof of the Theorem**

Let us assume the orbit $\gamma \cdot G$ is finite for each $\gamma \in L^+$. By the Frobenius Reciprocity Theorem, $L^2(L\backslash K, m) = \bigoplus_{\gamma \in L^+} L^2_\gamma(L\backslash K)$ is an orthogonal direct sum of its non-zero finite dimensional isotypic summands $L^2_\gamma(L\backslash K)$ indexed by $L^+$. Moreover, $L^2_\gamma(L\backslash K)$ is the image of the $\gamma$-primary projection $P_\gamma$ of $\pi$.

$P_\gamma$ is given by $\dim(\gamma) \int_K \chi_\gamma(k^{-1}\pi(k))dk$ where $\chi_\gamma(\cdot)$ is the character associated to $\gamma$.

(See [1], Theorem 5.10). By the cocycle identity, a short calculation shows that for each $(k, k_1, g)$ in $K \times K \times_s G$ we have for a.e. $\pi$

$$R(\pi, (k, g)) \pi(k_1) = \pi(k(g \cdot k_1)) R(\pi, (k, g)).$$
Then one can verify that $P_{γ,g^{-1}} R(ς,(k,g)) = R(ς,(k,g))P_γ$ holds for almost all $ς$, for each $(k,g,γ)$ in $K × s G × ̄K$ and $k_1 ∈ K$.

In particular, $R(ς,(k,g))L^2(Ł\backslash K) ⊆ L^2(Ł\backslash K)$ for a.e. $ς$, for each $(k,g,γ) ∈ K × s G × ̄K$. If $θ$ is a $G$-orbit in ̄$K$ then for each $(k,g) ∈ K × s G$, \( R(ς,(k,g)) \left( \bigoplus_{γ ∈ θ} L^2(Ł\backslash K) \right) = \bigoplus_{γ ∈ θ} L^2(Ł\backslash K) \) for a.e. $ς$ since $R(ς,(k,g))$ is a unitary operator for a.e. $ς$.

Hence, the trivial bundle $L^2(Ł\backslash K,m)$ over ̄$S$ is a direct sum of finite dimensional invariant Hilbert subbundles of the form $\left\{ \bigoplus_{γ ∈ θ} L^2(Ł\backslash K) : θ ∈ ̄S \right\}$. In particular, the extension $p : S → ̄S$ has relative discrete spectrum.

Conversely, suppose $p : S → ̄S$ is an extension with relative discrete spectrum. Thus, $L^2(Ł\backslash K,m)$ is a direct sum of finite dimensional invariant Hilbert subbundles $\{ V_i(ς) : θ ∈ ̄S \}$ over ̄$S$ for $i = 1, 2, ...$. Thus, for each $i$ and $k ∈ K$, $R(ς,k)V_i(ς) = V_i(ς)$ for ̄$S$-a.e. $ς$. Since the only connul Borel multiplicative subset of $K$ is $K$ itself then for ̄$S$-a.e. $ς$, $R(ς,k)V_i(ς) = V_i(ς)$ for all $i$ and $k ∈ K$. Hence, for ̄$S$-a.e. $ς$, the mapping $k → R(ς,k)$ is a unitary representation of $K$ on $V_i(ς)$ for all $i$. If $R(ς,k)$ defines a unitary representation of $K$ on $V_i(ς)$, let $Λ(i,ς) = \{ γ ∈ ̄K : V_i(ς) > γ \}$ where $V_i(ς) > γ$ if and only if the $γ$-isotypical summand in $V_i(ς)$ is non-zero. Otherwise, if $R(ς,k)$ does not define a representation on $V_i(ς)$ let $Λ(i,ς)$ be the empty set.

Fix $γ_0 ∈ L^1$. Since $\bigoplus_j V_j(ς) = L^2(Ł\backslash K) = \bigoplus_{γ ∈ L^2} L^2(Ł\backslash K)$ is an orthogonal direct sum for ̄$S$-a.e. $ς$, then we can choose an index $i_0$ such that the following set has positive ̄$S$-measure

$$\{ θ ∈ ̄S : k → R(ς,k) \text{ defines a representation of } K \text{ on } V_{i_0}(ς) \text{ and } γ_0 < V_{i_0}(ς) \}.$$  

Furthermore, this set with positive measure can be written as a countable union of sets indexed by the finite subsets $F$ of ̄$K$ containing $γ_0$, namely

$$\bigcup_{F \in F} \{ θ ∈ ̄S : k → R(ς,k) \text{ defines a representation of } K \text{ on } V_{i_0}(ς) \text{ and } Λ(i_0,ς) = F \}.$$  

Given $γ ∈ ̄K$ and $(k,g) ∈ K × s G$, $γ < V_0(ς)g$ if and only if $γ · g^{-1} < V_i(ς)$ for ̄$S$-a.e. $ς$ since $R(ς,(k,g))L^2(Ł\backslash K) = L^2(Ł\backslash K)$ for a.e. $ς$.

Define a Borel mapping $η$ from ̄$S$ into the finite subsets of ̄$K$ by $η(ς) = Λ(i_0,ς), θ ∈ ̄S$. Since $η$ is $G$-equivariant and ̄$S$ is a $G$-invariant measure then $(̄S ◦ η^{-1})$ is a $G$-invariant probability measure on the finite subsets of $̄K$. Choose a finite subset $F$ of ̄$K$ containing $γ_0$ such that

$$0 < ̄S \{ θ ∈ ̄S : k → R(ς,k) \text{ defines a representation of } K \text{ on } V_{i_0}(ς) \text{ and } Λ(i_0,ς) = F \}.$$  

Hence, $(̄S ◦ η^{-1})(F) > 0$ and the $G$-orbit of $γ_0$ in $̄K$ is finite. This completes the proof of the Theorem.
The measure $\mu$ on $S$ being invariant is crucial to our Theorem. To show this we exhibit an example of an extension $p : S \to \overline{S}$ with a quasiinvariant measure that is not invariant and for which the Theorem fails.

**Example:** The special linear group $SL(2, \mathbb{Z})$ with integer entries acts on the 2-torus $T^2$ by automorphisms. Namely, if $A \in SL(2, \mathbb{Z})$ and if we set $\exp(x, y) = (e^{2\pi ix}, e^{2\pi iy}) \in T^2$ then this action is defined by $A \cdot \exp(x, y) = \exp((x, y) \cdot A^t)$, where $(x, y) \in \mathbb{R}^2$ and $A^t$ is the transpose of $A$. With these we can form a semi-direct product group $T^2 \times_s SL(2, \mathbb{Z})$.

Each $(n, m) \in \mathbb{Z}^2$ defines a character of $T^2$ and this is given by $(n, m)\exp(x, y) = e^{2\pi i(nx+my)}$. $SL(2, \mathbb{Z})$ acts on the dual $\mathbb{Z}^2$ of $T^2$. In particular, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, $(n, m) \in \mathbb{Z}^2, x, y \in \mathbb{R}$ then $((n, m) \cdot A) \exp(x, y) \equiv (n, m)(A \cdot \exp(x, y))$. There are infinite $SL(2, \mathbb{Z})$-orbits in the dual of $T^2$ since a direct calculation shows $(n, m) \cdot A \equiv (an + cm, bn + dm) \in \mathbb{Z}^2$.

Let $\nu$ be a quasiinvariant probability measure on $SL(2, \mathbb{Z})$ i.e. $\nu(F) = 0$ if and only if $\nu(F \cdot A) = 0$ for any $A \in SL(2, \mathbb{Z}), F \subseteq SL(2, \mathbb{Z})$. Let $S$ be the measure space $T^2 \times SL(2, \mathbb{Z})$ with the product measure $\mu \times \nu$ where $\mu$ is the normalized Haar measure on $T^2$. If $(k, A) \in T^2 \times_s SL(2, \mathbb{Z})$ and $(k_1, w) \in S$ then we let $T^2 \times_s SL(2, \mathbb{Z})$ act on $S$ by

$$(k_1, w) \cdot (k, A) \equiv (A^{-1} \cdot (k_1)k, wA) \in S$$

where $A^{-1} \cdot (k_1)k$ is the action of $SL(2, \mathbb{Z})$ on $T^2$ and $wA$ is matrix multiplication. Let $\overline{\mathfrak{f}} = \{1\} \times SL(2, \mathbb{Z})$ and note that $\overline{\mathfrak{f}}$ meets each $T^2$-orbit in $S$ exactly once. Without loss of generality we may assume $\overline{\mathfrak{f}}$ is $SL(2, \mathbb{Z})$ and is equipped with the measure $\nu$.

We will show the extension $p : S \to \overline{S}$ given by $(k, w) \to w$ has relative discrete spectrum. Let $\mu \times \nu = \int_{\overline{\mathfrak{f}}} \mu \times \delta_w \; d\nu(w)$ be a disintegration of $\mu \times \nu$ over the fibers of $p$ where $\delta_w$ is the point-mass measure on $SL(2, \mathbb{Z})$ concentrated on $w$. One can show $(\mu \times \delta_w) \cdot A = \mu \times \delta_{wA}$ and $L = \{1, 1\}$ is the stabilizer in $T^2$ of each element of $\overline{\mathfrak{f}}$. Let $R$ be the cocycle representation associated to the extension $p$. Then for each $(k, A) \in T^2 \times_s SL(2, \mathbb{Z})$ and $w \in SL(2, \mathbb{Z})$, $R(w, (k, A)) : L^2(T^2, \mu) \to L^2(T^2, \mu)$ is a unitary operator given by

$$R(w, (k, A))h(k_1) = h(A^{-1} \cdot (k_1)k)$$

for each $h \in L^2(T^2, \mu), k_1 \in T^2$.

In particular, if $h = (n, m) \in \mathbb{Z}^2$ is an irreducible character of $T^2$ then a direct calculation shows that $R((1, w), (1, A))(n, m)$ is the irreducible character of $T^2$ defined by the matrix product $(n, m) \cdot A^{-1}$. Denote the complex linear span of the character $(n, m)$ of $T^2$ by $\mathfrak{c} : (n, m)$. For each $(n, m) \in \mathbb{Z}^2$, let us define a one dimensional invariant Hilbert bundle $\{H_{n,m}(w) : w \in \overline{\mathfrak{f}}\}$ over $\overline{\mathfrak{f}}$ by setting $H_{n,m}(w) = \mathfrak{c}((n, m) \cdot w)$. 

For any $A \in SL(2, \mathbb{Z})$,

$$L^2(T^2, \mu) = \bigoplus_{(n,m) \in \mathbb{Z}^2} C \cdot ((n,m) \cdot A)$$

is a well known orthogonal direct sum. Thus, the trivial bundle $L^2(T^2, \mu)$ over $\mathbb{S}$ is a direct sum of the Hilbert bundles $\{H_{n,m}(w) : w \in \mathbb{S}\}$ where the sum is indexed over $\mathbb{Z}^2$. Hence, the extension $p : S \to \mathbb{S}$ has relative discrete spectrum.

Finally, let us describe the ergodic actions of $K \times_s G$ which leave a probability measure invariant. Suppose $G \cdot \gamma$ is a finite orbit for each $\gamma \in L^\perp$. Then $S$ is essentially given by a skew-product action defined by a cocycle. (See [2], Theorem 4.3). In particular, there exist a compact group $M$, a closed subgroup $M_0$ of $M$, a cocycle $\beta : \mathbb{S} \times K \times_s G \to M$ with Mackey dense-range such that $S$ and $\mathbb{S} \times M_0 \backslash M$ are isomorphic $K \times_s G$-spaces. The $K \times_s G$-action on $\mathbb{S} \times M_0 \backslash M$ is given by $(\pi, M_0 m) \cdot (k,g) = (\pi \circ g, M_0 m \beta(\pi, (k,g))$ where $(\pi, M_0 m) \in \mathbb{S} \times M_0 \backslash M$, $(k,g) \in K \times_s G$.

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