On a Lie group with constant negative sectional curvature

Edgar Reyes
Department of Mathematics
Southeastern Louisiana University
Hammond, Louisiana, USA 70402
email: ereyes@selu.edu

Abstract

Let $\lambda > 0$ be a positive real number, and let $n \geq 1$ be an integer. Let $G = \mathbb{R}^n \times \mathbb{R}$ be a semi-direct product Lie group where the group multiplication in $G$ is defined by

$$(v_1, x_1) \ast (v_2, x_2) = (v_1 + e^{\lambda x_1} v_2, x_1 + x_2)$$

for all $v_i \in \mathbb{R}^n$, $x_i \in \mathbb{R}$, and $i = 1, 2$. We show $G$ has constant sectional curvature $-\lambda^2$, and describe the irreducible unitary representations of $G$.

2010 Mathematic Subject Classification: 22D10, 22E15
Keywords: Lie group, curvature, induced representation

1 Introduction

We construct a Lie group $G$ with constant negative sectional curvature. If the constant sectional curvature is $-1$, we construct an isometry between $G$ and the upper-half-space model for hyperbolic geometry. Also, we determine the irreducible unitary representations of $G$.

The Lie groups with negative sectional curvature have been classified by Ernst Heintze [3], which we state next.

**Theorem 1 (E. Heintze, 1974)** Let $\mathfrak{g}$ be a Lie algebra of a Lie group. If there exists an $x \in \mathfrak{g}$ such that

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathbb{R}x$$

and the eigenvalues of the restriction of $ad(x)$ to $[\mathfrak{g}, \mathfrak{g}]$ have positive real part, then the Lie group has negative sectional curvature. The converse is true.

In 1977, John Milnor constructs a special class of solvable Lie algebras with constant negative sectional curvature.
Theorem 2 (J. Milnor, see Example 1.7 in [5]) Let $\lambda > 0$ be a positive real number. Let $\mathfrak{g}$ be the Lie algebra of a Lie group such that

$$\mathfrak{g} = u + \mathbb{R}b$$

where $u$ is a commutative ideal of $\mathfrak{g}$, $b \in \mathfrak{g}$ is a unit vector that is orthogonal to $u$, and $[b, x] = \lambda x$ for all $x \in u$. Then the Lie group has constant negative sectional curvature $-\lambda^2$.

As far as we know, the group in Theorem 2 has not been constructed. We construct a group $G$ satisfying Theorem 2 as follows. Let $\lambda > 0$. Consider a semi-direct product group

$$G = \mathbb{R}^n \rtimes \mathbb{R}$$

where the group multiplication in $G$ is defined by

$$(v_1, x_1) \ast (v_2, x_2) = (v_1 + e^{\lambda x_1}v_2, x_1 + x_2)$$

for all $v_i \in \mathbb{R}^n$, $x_i \in \mathbb{R}$, and $i = 1, 2$. Since the group multiplication is differentiable, $G$ is a Lie group. Let $e = (0, 0) \in G$ denote the identity of the group $G$.

For any $g \in G$, let $T_g G$ denote the space of tangent vectors at $g$. Let $\xi$ be a differentiable function from an open interval containing zero such that $\xi(0) = g$. The derivative satisfies $\dot{\xi}(0) = (x', x_{n+1}) \in T_g G$ for some column vector $x' \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}$. We say $\xi$ is a differentiable curve in $G$ that passes through $g \in G$, and $v = (x', x_{n+1})$ is tangent to the curve $\xi$ at $g$. For the underlying sets, we have $T_g G = \mathbb{R}^n \times \mathbb{R}$. We regard the tangent vectors in $T_g G$ as column vectors. For $g \in G$, let $C^\infty(g)$ denote the space of real-valued functions that are differentiable on some open neighborhood of $g$. A tangent vector $v = \dot{\xi}(0) \in T_g G$ defines a real linear mapping from $C^\infty(g)$ into $\mathbb{R}$ such that if $f \in C^\infty(g)$, we let

$$v(f) = \frac{d}{dt} \bigg|_{t=0} [f \circ \xi(t)] \in \mathbb{R}.$$  \hfill (1.2)

Let $w^T$ denote the transpose of a column vector $w$. Let

$$\nabla(f(g)) = \left( \frac{\partial f}{\partial x_1} \bigg|_g, \ldots, \frac{\partial f}{\partial x_{n+1}} \bigg|_g \right) \in \mathbb{R}^{n+1}$$

denote the gradient of $f$ at $g$. Then by the chain rule

$$v(f) = \nabla(f(g))w^T.$$  \hfill (1.3)

Let $\xi$ be a differentiable curve that passes through the identity $e \in G$. If $\xi(s+t) = \xi(s)\xi(t)$ for all $s, t \in \mathbb{R}$, we say $\xi$ is a 1-parameter subgroup of $G$. We describe the 1-parameter subgroups of $G$.

Lemma 3 Let $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}$. Let $\xi : \mathbb{R} \to G$ be a 1-parameter subgroup of $G$ satisfying $\dot{\xi}(0) = (x_0, y_0)$. If $y_0 = 0$, then $\xi(t) = (tx_0, 0)$. While if $y_0 \neq 0$, then

$$\xi(t) = \left( e^{\lambda y_0} - \frac{1}{\lambda y_0} x_0, ty_0 \right).$$  \hfill (1.4)
Proof If \( y_0 = 0 \), then \( \xi(t) = (tx_0, 0) \) is a 1-parameter subgroup satisfying \( \dot{\xi}(0) = (x_0, 0) \). Suppose \( y_0 \neq 0 \) and \( \xi(t) = (\psi(t), ty_0) \) for some differentiable function \( \psi : \mathbb{R}^n \rightarrow \mathbb{R}^n \). The group multiplication in \( G \) implies

\[
(\psi(s + t), (s + t)y_0) = (\psi(s) + e^{\lambda y_0} \psi(t), (s + t)y_0).
\]

When \( t \neq 0 \), we find

\[
\frac{\psi(s + t) - \psi(s)}{t} = e^{\lambda y_0} \psi(t).
\]

As \( t \to 0 \), we obtain \( \dot{\psi}(s) = e^{\lambda y_0} x_0 \). Then integrating and applying \( \xi(0) = e \), we obtain (1.4).

\[ \square \]

2 Lie brackets

We review some standard facts about left-invariant vector fields on Lie groups, specially as they apply to \( G \) [1][2][5]. Let \( z \in T_e G \) be a tangent vector. We define a left-invariant vector field \( Z \) on \( G \) in the following way. If \( g \in G \), let \( L_g : G \rightarrow G \) be left multiplication by \( g \), i.e., \( L_g(x) = gx \), \( x \in G \). Let \( dL_g : T_e G \rightarrow T_g G \) be the differential of \( L_g \) at the identity \( e \in G \). We set \( Z(g) = dL_g(z) \in T_g G \). Clearly, \( Z(e) = z \). If \( w \in T_e G \) is another tangent vector, let \( W \) be the left-invariant vector field on \( G \) such that \( W(e) = w \). Then \( ZW - WV \) defines a left-invariant vector field on \( G \) which we denote by \([Z,W]\). We set

\[
[z, w] = [Z,W](e) \in T_e G
\]

which is the value of the left-invariant vector field \([Z,W]\) at \( e \in G \). The space of left invariant vector fields on \( G \) is the Lie algebra, \( \mathcal{L}(G) \), of \( G \).

The mapping \( Z \in \mathcal{L}(G) \mapsto Z(e) \in T_e(G) \) is a one-to-one correspondence between the left invariant vector fields on \( G \) and the tangent vectors at the identity \( e \in G \). For each \( k \in \{1, \ldots, n + 1\} \), let \( e_k \in T_e G = \mathbb{R}^{n+1} \) be the standard unit column vector that has a 1 in the \( k \)-th entry, and where all other entries are zeros. Let \( X_k \in \mathcal{L}(G) \) be the unique left-invariant vector field on \( G \) such that \( X_k(e) = e_k \). Applying identity (1.2), if \( \gamma(t) = te_k \in G \), \( t \in \mathbb{R} \), then

\[
e_k(f) = \frac{d}{dt} \bigg|_{t=0} [f \circ \gamma(t)] = \frac{\partial f}{\partial x_k} \bigg|_e.
\]

Lemma 4 For \( k \in \{1, \ldots, n + 1\} \), let \( X_k \in \mathcal{L}(G) \) be the left-invariant vector fields on \( G \) such that \( X_k(e) = e_k \). Let \( g = (g', g_{n+1}) \in G \) where \( g' \in \mathbb{R}^n \) and \( g_{n+1} \in \mathbb{R} \). Then

\[
X_k(g) = \begin{cases} 
 e^{\lambda g_{n+1}} \frac{\partial}{\partial x_k} & \text{if } 1 \leq k \leq n \\
\frac{\partial}{\partial x_{n+1}} & \text{if } k = n + 1.
\end{cases}
\]
Proof: If \( f \in C^\infty(g) \), then
\[
X_k(g)f = dL_g(e_k)f = \frac{\partial}{\partial x_k}(f \circ L_g).
\]
However, if \( x = (x', x_{n+1}) \in G \) where \( x' \in \mathbb{R}^n \), \( x_{n+1} \in \mathbb{R} \), then
\[
(f \circ L_g)(x) = f(g \ast x) = f\left(g' + e^\lambda g_{n+1} x', g_{n+1} + x_{n+1}\right).
\]
Applying the chain rule, we obtain the conclusion of the theorem.

In the next lemma, we describe the Lie brackets in \( \mathcal{L}(G) \).

Lemma 5: For \( k \in \{1, \ldots, n+1\} \), let \( X_k \in \mathcal{L}(G) \) be the left-invariant vector fields on \( G \) such that \( X_k(e) = e_k \). If \( 1 \leq i, j \leq n \), then \( [X_i, X_j] = 0 \) and \( [X_{n+1}, X_j] = \lambda X_j \).

Proof: Let \( x' \in \mathbb{R}^n \), \( x_{n+1} \in \mathbb{R} \), and let \( x = (x', x_{n+1}) \in G \). Applying Lemma 4,
\[
X_j(x)f = p(x)\frac{\partial f}{\partial x_j}_e.
\]
where
\[
p(x) = e^{\lambda x_{n+1}}.
\]
Note, \( \frac{\partial}{\partial x_i} p(x) = 0 \) since \( i \neq n + 1 \). By Lemma 4 and the product rule,
\[
X_i(g)(X_j(f)) = p(g)\frac{\partial}{\partial x_i} \left[p(x)\frac{\partial f}{\partial x_j}\right] = p(g)^2 \frac{\partial}{\partial x_i} \left[p(x)\frac{\partial f}{\partial x_j}\right].
\]
Since \( \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} \), we find
\[
X_i(g)(X_j(f)) = X_j(g)(X_i(f)).
\]
Then \( [X_i, X_j] = 0 \). Similarly, by the product rule, we find
\[
X_{n+1}(g)(X_j(f)) = \frac{\partial}{\partial x_{n+1}} \left[p(x)\frac{\partial f}{\partial x_j}\right] = \lambda p(g) \frac{\partial f}{\partial x_j}_g + p(g) \frac{\partial}{\partial x_{n+1}} \left[p(x)\frac{\partial f}{\partial x_j}\right].
\]
Also, by Lemma 4,
\[
X_j(g)(X_{n+1}(f)) = p(g) \frac{\partial}{\partial x_j}_g \left[\frac{\partial f}{\partial x_{n+1}}\right].
\]
Thus,
\[
X_{n+1}(g)(X_j(f)) - X_j(g)(X_{n+1}(f)) = \lambda p(g) \frac{\partial f}{\partial x_j}_g = \lambda X_j(g)(f).
\]
Hence, \( [X_{n+1}, X_j] = \lambda X_j \).

Now, by applying the correspondence in (2.5) and Lemma 5, the Lie bracket in the tangent space \( T_eG \) can be described.

Corollary 6: Let \( e_k \in T_eG = \mathbb{R}^n \times \mathbb{R} \) be the usual basis vectors where \( 1 \leq k \leq n + 1 \). If \( 1 \leq i, j \leq n \), then \( [e_i, e_j] = 0 \) and \( [e_{n+1}, e_j] = \lambda e_j \).
3 Riemannian manifold

We assign a left-invariant Riemannian metric on $G$ as carried out by John Milnor in [5]. Let $X_k \in \mathcal{L}(G)$, $1 \leq k \leq n+1$, be the basis of left-invariant vector fields in Lemma 5. Let $g \in G$ and $1 \leq i, j \leq n+1$. We assign a positive-definite symmetric bilinear form $\langle \cdot, \cdot \rangle_g$ on the tangent space $T_g G$ satisfying $\langle X_i(g), X_j(g) \rangle_g = \delta_{ij}$. In other words, the left-invariant vector fields $X_k$ are orthonormal in this chosen Riemannian metric. Then the function $p$ from (2.6) satisfies

$$
\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_g = \begin{cases} 
\frac{1}{p^2(g)} \delta_{ij} & \text{if } 1 \leq i, j \leq n \\
1 & \text{if } i = j = n+1.
\end{cases}
$$

(3.7)

Adopting the notations of Theorem 2, let $\mathfrak{g} = \mathcal{L}(G)$, let $\mathfrak{u}$ be the commutative ideal spanned by $\{X_1, \ldots, X_n\}$, and let $b = X_{n+1}$. Then $\mathcal{L}(G)$ satisfies Theorem 2.

**Lemma 7** Let $\lambda > 0$, and let $G$ be the Lie group in (1.1). Then the Lie algebra of $G$ satisfies Theorem 2, and $G$ has constant sectional curvature $-\lambda^2$.

If $\lambda = 1$, $G$ has constant sectional curvature of $-1$. The upper-half space

$$
\mathcal{H}^{n+1} = \{(x', x_{n+1}) : x' \in \mathbb{R}^n, x_{n+1} > 0\}
$$

is a model for hyperbolic geometry. If $\lambda = 1$, we establish a Riemannian isometry between $G$ and $\mathcal{H}^{n+1}$. Let $x = (x', x_{n+1}) \in \mathcal{H}^{n+1}$. A Riemannian metric $\langle \cdot, \cdot \rangle_x^{hyp}$ for the tangent space $T_x \mathcal{H}^{n+1}$ is given by

$$
\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle^{hyp}_x = \delta_{ij} \frac{1}{x_{n+1}^2}
$$

(3.8)

for $1 \leq i, j \leq n+1$. In the next lemma, let $\ln(w)$ be the natural logarithm of $w > 0$.

**Lemma 8** Let $\lambda = 1$. Let $\phi : \mathcal{H}^{n+1} \to G$ be a function defined by $\phi(x', x_{n+1}) = (x', \ln(x_{n+1}))$ where $(x', x_{n+1}) \in \mathcal{H}^{n+1}$. Then $\phi$ is an isometry of Riemannian manifolds.

**Proof** The proof is a standard verification. Clearly, $\phi$ is differentiable and bijective. Let $x = (x', x_{n+1}) \in \mathcal{H}^{n+1}$, and let $d\phi : T_x \mathcal{H}^{n+1} \to T_{\phi(x)} G$ be the differential of $\phi$ at $x$. Let $\phi(x) = (\phi_1(x), \ldots, \phi_{n+1}(x))$ be expressed in component form where $\phi_i(x) = x_i$, $1 \leq i \leq n$, and $\phi_{n+1}(x) = \ln(x_{n+1})$. Let $1 \leq i, j, \leq n+1$ be indices. Let $D[\phi]$ be the Jacobian matrix for $\phi$ at $x$, i.e., $D[\phi]$ is a square matrix whose $(i, j)$-entry is $\frac{\partial \phi_j}{\partial x_i} |_x$. Then $D[\phi] = I_n \oplus \frac{1}{x_{n+1}}$ is a block matrix where $I_n$ is the $n$-by-$n$ identity matrix.

Let $v \in T_x \mathcal{H}^{n+1}$ be a tangent vector. Let $\xi$ be a curve in $\mathcal{H}^{n+1}$ such that $\xi(0) = x$ and $\xi(0) = v$. By definition, the image of $f \in C^\infty(\phi(x))$ under $d\phi(v)$ is denoted and given by

$$
d\phi(v)(f) = \frac{d}{dt} \bigg|_{t=0} f \circ \phi \circ \xi(t).
$$
Let \( \xi(t) = (\xi_1(t), \ldots, \xi_{n+1}(t)) \), and \( \dot{\xi}(0) = (v_1, \ldots, v_{n+1}) \in \mathbb{R}^{n+1} = T_x\mathcal{H}^{n+1} \) be column vectors. Applying the chain rule,
\[
d\phi(v)(f) = \sum_{i=1}^{n+1} \frac{\partial (f \circ \phi)}{\partial x_i} \dot{\xi}_i(0)
\]
\[
= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} \frac{\partial f}{\partial x_j} \frac{\partial \phi_j}{\partial x_i} \dot{\xi}_i(0)
\]
\[
= \sum_{j=1}^{n+1} \frac{\partial f}{\partial x_j} \sum_{i=1}^{n+1} \frac{\partial \phi_j}{\partial x_i} \dot{\xi}_i(0).
\]

Let \( w^T \) denote the transpose of a matrix \( w \). Since \( \dot{\xi}(0) = v \), we obtain
\[
d\phi(v)(f) = \nabla f(\phi(x)) D[\phi]v^T.
\]

Next, we apply (1.3). The right side of the above equation is the image of \( f \) under the tangent vector \( vD[\phi]^T \in T_{\phi(x)}G \). Then \( d\phi(v) = vD[\phi]^T \). Since \( D[\phi] = I_n \oplus \frac{1}{x_{n+1}} \), we find \( d\phi(v) = (v_1, \ldots, v_n, \frac{1}{x_{n+1}} v) \). Since \( \lambda = 1 \), we have \( p(\phi(x)) = x_{n+1} \). Applying the Riemannian metrics (3.7) and (3.8), we find
\[
\langle d\phi(v), d\phi(v) \rangle_{\phi(x)} = \frac{1}{x_{n+1}^2}(v_1^2 + \cdots + v_{n+1}^2) = \langle v, v \rangle_{x}^{hyp}.
\]

Hence, \( \phi \) is an isometry. \( \Box \)

4 Induced representations

We review some constructions from George Mackey’s theory of induced unitary group representations [4, page 135]. Let \( M \) be a locally compact, Hausdorff, second countable topological group. Let \( H \leq M \) be a closed subgroup of \( M \). Let \( \mathcal{H} \) be a separable complex Hilbert space, and let \( \mathcal{U}(\mathcal{H}) \) denote the group of unitary operators on \( \mathcal{H} \). Let \( L : H \to \mathcal{U}(\mathcal{H}) \) be a group homomorphism denoted by \( L(h) = L_h \in \mathcal{U}(\mathcal{H}) \), \( h \in H \). Suppose for all \( v \in \mathcal{H} \), the function \( h \mapsto L_h(v) \) is continuous from \( H \) into \( \mathcal{H} \). We say \( L \) is a unitary group representation of \( H \) on \( \mathcal{H} \).

Let \( M/H \) be the set of right cosets. Let \( m_1, m_2 \in M \) and \( Hm_1, Hm_2 \in M/H \). The group \( M \) acts on \( M/H \) as described by a mapping from \( M/H \times M \) to \( M/H \) where \((Hm_1, m_2) \mapsto Hm_1m_2 \). If \( E \subseteq M/H \) is a Borel subset, let \( Em_2 = \{sm_2 : s \in E \} \). There is a quasi-invariant measure \( \mu \) on \( M/H \) that is not identically zero such that \( \mu(E) = 0 \) if and only if \( \mu(Em_2) = 0 \). Let \( \rho_x : G/H \to \mathbb{R}^+ \) be a positive Borel function such that \( \mu_x(E) = \int_E \rho_x(s)d\mu(s) \) for all Borel subsets \( E \subseteq M/H \).

Let \( \langle \cdot, \cdot \rangle_{\mathcal{H}} \) denote the inner-product in the complex separable Hilbert space \( \mathcal{H} \). Let \( \|v\|_{\mathcal{H}} = \langle v, v \rangle_{\mathcal{H}}^{1/2}, v \in \mathcal{H} \). Let \( \mathcal{K}' \) be a space of functions \( f : M \to \mathcal{H} \) such that

(a) For all \( v \in \mathcal{H} \), the function \( h \mapsto \langle f(h), v \rangle_{\mathcal{H}} \) is continuous on \( H \), and
(b) \( f(hm) = L_h(f(m)) \) for all \( h \in H, m \in M \).

If \( f_1, f_2 \in \mathcal{K}' \), then \( \langle f_1(hx), f_2(hx) \rangle_{\mathcal{H}} \) is constant and independent of \( h \in H \) for all \( x \in M \) due to part (b). We consider the following inner-product in \( \mathcal{K}' \)

\[
\langle f_1, f_2 \rangle = \int_{M/H} \langle f_1(s), f_2(s) \rangle_{\mathcal{H}} d\mu(s).
\]

Then \( \mathcal{K} = \{ f \in \mathcal{K}' : \langle f, f \rangle < \infty \} \) is a complex separable Hilbert space with the inner-product from \( \mathcal{K}' \). Let \( x, y \in M \) and \( f \in \mathcal{K} \). Let \( \mu^L : M \to \mathcal{U}(\mathcal{K}) \) be a unitary representation of \( M \) denoted by \( \mu^L_x \in \mathcal{U}(\mathcal{K}) \) and defined by

\[
(\mu^L_x f)(y) = \sqrt{\rho_x(y)} f(yx).
\]

If \( \nu \) is also a quasi-invariant measure \( M/H \), then \( \mu^L \) and \( \mu^L \) are equivalent unitary representations of \( M \). We omit the dependence on \( \mu \). We say \( U^L \) is the unitary representation of \( M \) that is induced from the unitary representation \( L \) of \( H \).

We determine the irreducible unitary representations of the group \( G \) (1.1). Let

\[
K = \{(x', 0) \in G : x' \in \mathbb{R}^n \}.
\]

(4.9)

If \( x = (x', 0) \in K \), and \( g = (g', g_{n+1}) \in G \), then

\[
g \ast x \ast g^{-1} = (e^{\lambda g_{n+1}} x', 0).
\]

(4.10)

Clearly, \( K \leq G \) is an abelian normal subgroup of \( G \). We consider the action of \( G \) on \( K \) by inner-conjugation (4.10). The \( G \)-orbit of \( x \in K \) is denoted and satifies

\[
\mathcal{O}(x) = \{(e^{\lambda g_{n+1}} x', 0) : g \in G \}.
\]

The stabilizer subgroup of \( x \in K \) is given by

\[
G_x = \{g \in G : e^{\lambda g_{n+1}} x' = x' \}.
\]

Lemma 9 Let \( x' \in \mathbb{R}^n \), and let \( x = (x', 0) \in K \). Then \( \mathcal{O}(x) = \{(\alpha x', 0) : \alpha > 0 \} \). If \( x' \neq 0 \), then \( G_x = K \). If \( x' = 0 \), then \( G_x = G \).

Let \( \hat{K} \) denote the group of characters of \( K \). For \( x = (x', 0) \in K \), let \( (x')^T \) denote the transpose of column vector \( x' \in \mathbb{R}^n \). Let \( exp \) denote the natural exponential function. Let \( L_x \in \hat{K} \) be a character of \( K \) such that if \( y = (y', 0) \in \hat{K} \) then

\[
L_x(y) = \exp(i(x')^T y').
\]

(4.11)

Let \( G \) act on \( \hat{K} \) as follows. For \( g \in G \), let \( (L_x)^g \in \hat{K} \) be defined by

\[
(L_x)^g(y) = L_x(g \ast y \ast g^{-1}).
\]

Applying (4.10), we find

\[
(L_x)^g = L_{g \ast x \ast g^{-1}}.
\]
We denote the $G$-orbit of $L_x$ by $O(L_x)$. The action of $G$ on $\hat{K}$ mirrors the action (4.10) of $G$ on $K$. Then $O(L_x) = \{L_k : k \in O(x)\}$. We denote the stabilizer of $L_x$ in $G$ by

$$\text{stab}(L_x) = \{g \in G : (L_x)^g = L_x\}.$$ 

Then $\text{stab}(L_x) = G_x$ for all $x \in K$.

For $x' \in \mathbb{R}^n$, let $\|x'\|$ denote the Euclidean norm of $x'$. Consider the closed (necessarily, Borel) subset of $K$:

$$S = \{(x', 0) \in K : \|x'\| = 1 \text{ or } x' = 0\}.$$ 

Note, $S$ intersects each $G$-orbit in $K$ exactly once. Then the quotient Borel structure on the space of $G$-orbits in $\hat{K}$ is countable separated, i.e., $K$ is regularly embedded in $G$ [4, page 186]. Consequently, by Mackey’s [4, Theorem 3.12], every irreducible unitary representation of $G$ is induced. We describe an elementary application of Theorem 3.12 to $G$.

**Theorem 10** Let $O$ be a $G$-orbit in $\hat{K}$, and let $L \in O$ be a character of $K$. Let $H_o$ be the stabilizer of $L$ in $G$.

(a) If $H_o = K$, then $U^L$ is an irreducible unitary representation of $G$.

(b) Assume $H_o = G$. If $t \in \mathbb{R}$, then $V_t(g) = \exp(itg_{n+1})$, $g = (g', g_{n+1}) \in G$, defines a character of $G$.

Up to equivalence of representations, every irreducible unitary representation of $G$ is obtained from either (a) or (b), and some $G$-orbit $O$ in $\hat{K}$.

**Proof** Since $L$ is a character of $K$, we write $L = L_x$ for some $x \in K$ (4.11). Since $H_o$ is the stabilizer subgroup of $L_x$, it follows from Lemma 9 that $H_o = K$ if and only if $x \neq 0$. If $x \neq 0$, then the induced unitary representation $U^{L_x}$ of $G$ is irreducible [4, Theorem 3.12].

Similarly, $H_o = G$ if $x = 0$. If $x = 0$, then $L = L_0$ is the identity one-dimensional character of $K$. Clearly, the group $G/K$ is isomorphic to $\mathbb{R}$, and we write $G/K \simeq \mathbb{R}$. The characters of $G/K \simeq \mathbb{R}$ may be extended to define characters of $G$ as follows. Let $g = (g', g_{n+1}) \in G$. For each $t \in \mathbb{R}$, let $V_t$ be a character of $G$ defined by

$$V_t(g) = \exp(itg_{n+1}).$$

Hence, that every irreducible representation (up to equivalence) of $G$ is obtained from either (a) or (b), and some $G$-orbit in $\hat{K}$ is an application of [4, Theorem 3.12].

**Corollary 11** A complete set of inequivalent irreducible unitary representation of $G$ is the set

$$\hat{G} = \{V_t : t \in \mathbb{R}\} \cup \{U^{L_x} : x \in S\}.$$
References


