Magnetic Fields in Matter

Introduction

We have seen that separating charges creates an electric field that in turn exerts a force on other charges. In the same way, MOVING charges (i.e., a current) creates a magnetic field that exerts forces on other currents. So it seems straightforward that magnetic fields should exert forces on matter. Matter is made up of electrons in orbit around nucleons. This orbital motion constitutes a current which will feel forces from a magnetic field. Further, nucleons are composed of smaller charges particles called quarks which are in motion inside the nucleus. These motions create currents which feel forces from magnetic fields (the magnetic forces on nucleons are the physical phenomena that make nuclear magnetic resonance and magnetic resonance imaging systems work). Since charge is conserved (and it is so incredibly hard to store up a reasonable amount of charge) it necessarily follows that sustained currents will form loops (why?). So it is reasonable to consider the effect of a magnetic field upon a current loop as a prelude to considering the effect of magnetic fields on matter.

Your textbook shows that the net force on a current loop in a uniform magnetic field is zero. This makes sense: since the current must first go one way and then the other (to form a loop) the magnetic force on one part of the loop will be in a direction opposite to that exerted on the other part of the loop. If the magnitude of the field is the same for all parts of the loop, then the magnitude of the forces on different parts will be the same. The same magnitude in opposite directions results in zero net force. These forces do exert torques as indicated in your textbook. But note that the argument above fails if the field changes significantly over the extent of the current loop. That is, if the field is larger in one part of the loop, that part of the loop will feel a stronger force than other parts, and so the net force will no longer be zero.

Refer to Problem # 94 in your textbook and you see that the force on a current loop is given by:

\[
\mathbf{F} = \mu_x \frac{dB_x}{dx} \mathbf{i} + \mu_y \frac{dB_y}{dy} \mathbf{j} + \mu_z \frac{dB_z}{dz} \mathbf{k}
\]

which can be written as:

\[
\mathbf{F} = \nabla (\mu \cdot \mathbf{B})
\]

where \(\mu\) is the magnetic moment of the current loop. In fact, ANY current which produces a magnetic moment in a nonuniform magnetic field will feel a force given by Equation 2. In this lab, we will create a nonuniform magnetic field and measure the force it exerts on a steel ball bearing. In the following section, analysis of this force will allow us to extract information about the material of which the bearing is made.

Magnetic forces on matter

While it is impossible calculate the magnetic moment of the electrons and nucleons comprising the ball bearing, it is possible to make some progress by adopting a phenomenological approach. By this it is meant that we make certain assumptions about the behavior of the matter and determine if our observations are consistent with those assumptions. The first assumption is that I can associate with each small volume
of the matter, \( dV \), a magnetic moment, \( d\mu \). This magnetic moment per unit volume is called the magnetization of the material:

\[
M = \frac{d\mu}{dV}
\]  

(3)

With considerable difficulty, one can show that a magnetized material (i.e., one that has a nonzero magnetization) in an external field feels a force given by:

\[
F = -\oint (\nabla \cdot M)BdV + \oint (M \cdot \hat{n})BdA
\]  

(4)

where \( B \) is the external field and the integrals are over the volume and surface of the magnetized material (see Classical Electrodynamics, 2nd Ed., J. D. Jackson, p. 207).

One final approximation is that the material is linear; that is, we assume that the magnetization is proportional to the applied field:

\[
M = \frac{\chi_m}{\mu_0} B
\]  

(5)

where \( \mu_0 \) is the permeability of free space and \( \chi_m \) is the susceptibility of the material (it is a measure of how easily the external field aligns and/or creates magnetic moments in the material - you should be able to show that \( \chi_m \) is a unitless number). With this, the force in Equation 4 can be written:

\[
F = -\frac{\chi_m}{\mu_0} \oint (\nabla \cdot B)BdV + \frac{\chi_m}{\mu_0} \oint (B \cdot \hat{n})BdA
\]

(6)

One of the Maxwell’s laws of electromagnetism is that \( \nabla \cdot B = 0 \) always, so the force simplifies to:

\[
F = +\frac{\chi_m}{\mu_0} \oint (B \cdot \hat{n})BdA
\]  

(7)

This is as far as we can go without a specific description of the experimental geometry. Further, in order to make the problem tractable, we will make several idealizations of the actual geometry to simplify the calculation.

The experiment will consist of placing a steel bearing near the opening of a solenoid. As seen in your textbook, the field near the end of a solenoid varies strongly with distance. (Some thought about equation 7 makes it clear that for there to be a net force on the ball bearing, it is necessary that the field vary over the bearing). With great difficulty, one can construct a spherical coordinate system centered on the ball bearing and write eq. 7 as:

\[
F_z = +\frac{\chi_m}{\mu_0} \rho^2 \int \int \int \left[ B(z) \right]^2 \sin \Phi \cos \Phi \ d\Phi \ d\Theta
\]  

(8)

where \( \rho \) is the radius of the bearing, \( z \) is the distance measured along the axis of the solenoid (where \( B \) is taken to point in the \( z \) direction). Note that reversing the direction of the magnetic field (\( B \rightarrow -B \)) makes no difference in the direction of the force - it is toward the center of the solenoid. The integral over \( \Theta \) is easy enough - it gives \( 2\pi \).

But the integral over \( \Phi \) is quite difficult, since the variation of the field with \( z \) is quite intractable (Eq. 29-8 of your textbook). However, we can simplify things by expanding the field near the end of the solenoid in a Taylor series expansion:

\[
B(z) \equiv B(0) + B'(0)z = B(0) \left[ 1 + \frac{B'(0)}{B(0)} z \right] = B(0) \left[ 1 + \frac{B'(0)}{B(0)} \rho \cos \Phi \right] = B(0) \left[ 1 + K \cos \Phi \right]
\]

(9)
Putting this in Eq 8 (and using $B(0) = 1/2 \mu_0 n I$), we have:

$$F_z = 2\pi \frac{\chi_m}{\mu_0} \rho^2 \left( \frac{1}{2} \mu_0 n I \right) \int_{0}^{\pi} \left[ 1 + K \cos \Phi \right] \sin \Phi \cos \Phi \, d\Phi$$

which is easily transformed to:

$$F_z = 2\pi \frac{\chi_m}{\mu_0} \rho^2 \left( \frac{1}{2} \mu_0 n I \right) \int_{-1}^{1} \left[ 1 + Ku \right] u \, du$$

The integral is straightforward, and results in a force of:

$$F_z = \frac{4\pi}{3} K\chi_m \mu_0 \rho^2 (nI)^2$$

Aside: We can look at this from another point of view. Consider a small bit of the bearing, $dV$, and suppose it has a magnetic moment of $d\mu$. In a nonuniform magnetic field, it will feel a force

$$dF = \nabla (\mu \cdot B)$$

and the bearing will then feel a force per unit volume of

$$\frac{dF}{dV} = \nabla (M \cdot B)$$

If we again assume that the field point along $z$ and only varies with $z$, and that the material is linear, the force per unit volume is

$$\frac{dF_z}{dV} = \frac{\chi_m}{\mu_0} \frac{d}{dz} \left( B(z) \right)^2$$

and the total force is

$$F_z = \frac{\chi_m}{\mu_0} \int \int \int \frac{d}{dz} \left( B(z) \right)^2 \, dV$$

We write the volume element as $dV = A(z) \, dz$, and approximate the $z$-dependent area as a constant $\pi \rho^2$, and get

$$F_z = \pi \rho^2 \frac{\chi_m}{\mu_0} \int \frac{d}{dz} \left( B(z) \right)^2 \, dz = \pi \rho^2 \frac{\chi_m}{\mu_0} \int d\left( B(z) \right)^2$$

The integral is trivial and the force is

$$F_z = \pi \rho^2 \frac{\chi_m}{\mu_0} \left( B(z) \right)^2 = \pi \rho^2 \mu_0 \chi_m (nI)^2$$

which differs from Eq 12 only in a numerical constant (of order 1). That is, we have obtained the correct expression for the force using a model of dubious justification. This is an example of what Professor Deering used to say: "Physicists have a knack for jumping into mathematical cesspools and coming out smelling like a rose."

**Conclusions**

The experiment consists of placing the bearing on a scale, positioning a solenoid above it and reading the force on the scale as the current through the solenoid is increased. How should you plot your data to allow you to verify or refute the expression for the force given by Eq 12? If it is verified, how can you extract the value of the susceptibility (with error bar)? How best do you make the world safe for democracy?