Resistive / Inductive / Capacitive Circuits.

Introduction

When a voltage source delivers energy to a circuit, we know that only two things can happen: the energy can be stored as either an electric or magnetic field (in a capacitor or inductor, respectively) or it can be dissipated as “heat” (in a resistor). This is quite significant, for it means that ANY circuit can be represented quite simply as a resistance, R, an inductance, L, and a capacitance, C. But note well: the MATH lets you conveniently separate these three elements. It may be much more difficult to separate them physically. For example, an inductor may consist of many loops of wire (i.e., a solenoid) which will give an inductance. But all that wire also provides a resistance that cannot be separated from the inductance. There are two mathematical methods to analyze an RLC circuit. We’ll do both.

A) “Equation of Motion”

I call this method the “equation of motion” because when we get done, we’ll see that there is a strong analogy between the behavior of an RLC circuit and a damped harmonic oscillator. That is, when we write an equation describing the RLC circuit voltages, it will be mathematically identical to Newton’s equation of motion for a damped harmonic oscillator. So, in a sense, Kirchoff’s rule is the “equation of motion” for an electrical circuit.

If we write Kirchoff’s loop rule for a DC voltage in series with a resistor, and inductor and a capacitor, we have:

\[ V - IR - L \frac{\Delta I}{\Delta t} - \frac{Q}{C} = 0 \]  

(1)

where, as always, I is the current, Q is the charge on the capacitor, V is the voltage provided by the battery and R, L, and C are the resistance, inductance and capacitance, respectively. Remember that \( I = \frac{\Delta Q}{\Delta t} \). Then we can rewrite Equation (1) as:

\[ V - \frac{\Delta Q}{\Delta t} R - \frac{Q}{C} = L \frac{\Delta I}{\Delta t} \]  

(2)

Not much of a difference, but now remember what the equation of motion for a damped harmonic oscillator looks like; that is, write Newton’s second law for a system consisting of a mass, m, on a spring of stiffness, k, with a drag force proportional to the speed:

\[ W - \frac{\Delta x}{\Delta t} b - kx = m \frac{\Delta v}{\Delta t} \]  

(3)

where you recall that \( v = \Delta v/\Delta t \). In Equation (3), W is the constant force applied to the mass (its weight), b is the drag coefficient and k is the stiffness of the spring. We can see here that there is NO mathematical difference between these two systems (changing from one to the other is ONLY a change of symbols). These two equations describe systems that are mathematically identical. I do NOT claim that the two systems are PHYSICALLY identical, merely that the two systems BEHAVE the same. So an RLC circuit in series with a battery should oscillate in the same way that a mass on a spring does. And the oscillations should damp away as the drag force
dissipates the energy. Finally, let’s compare Equations (2) and (3) to find the analogies between the spring system and the electrical system. We can see that the voltage $V$ plays the same role as the constant force, $W$. The resistance plays the role of the drag force. This makes physical sense because a drag force dissipates energy in the same way that a resistor does. The stiffness of the spring corresponds to the inverse of the capacitance (i.e., $k$ corresponds to $1/C$). This is also sensible, as a small $k$ means that a large displacement is necessary for a large force; likewise, a small $1/C$ (i.e., a large capacitance) means that a large charge is necessary for a large voltage.

Finally, the inductance plays the role of the inertia (or mass). That is, the larger the mass on a spring, the more that system resists a change in the velocity. Similarly, the larger the inductance, the more the system resists any change in the current. Finally, the charge $Q$ plays the role of the position, $x$. With these analogies, we can consider this system from the perspective of energy.

**B) Energy**

In mechanics lecture class, you’ve discussed the energy of a harmonic oscillator. There is a kinetic energy, $\frac{1}{2}mv^2$, and a stored elastic energy, $\frac{1}{2}kx^2$. The total energy is then:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E_{tot}$$

(4)

So even if we didn’t already know, using the analogies from above we could guess that the energy of an electrical circuit is:

$$\frac{1}{2}LI^2 + \frac{1}{2}\frac{Q^2}{C} = E_{tot}$$

(5)

But we DO already know: the first term is the energy stored in the magnetic field created by the inductor, and the second term is the energy stored in the electric field created by the capacitor. If there is no drag force, the total energy stays the same, oscillating from kinetic to potential (in the case of the spring) and between magnetic and electric (in the case of the inductor and capacitor). But we know that oscillations DO eventually cease (start a pendulum oscillating and come back the next day – will it still be moving?). Likewise, any electrical circuit will have some resistance that dissipates energy. In the case of the spring, the energy lost will be the work done by the drag force – Energy lost = work = force x distance = $-bv \Delta x$. In the case of the electrical system, the energy lost will be the energy dissipated by the resistor: Energy lost = Power x time = $-I^2R \Delta t = -IR (I\Delta t) = -IR \Delta Q$. So we see that the analogy between the two systems is complete, whether considering the equations of motion or the energy. In this lab, we will set up and analyze an RLC circuit and in the process, determine the “mass” of the electrical circuit.

The experiment will consist of setting up an oscilloscope to read the voltage across the capacitor. This is equivalent to measuring the charge and therefore the “distance” of the oscillator. The circuit will start disconnected from the battery (zero “weight”) and then the battery will be connected (the “weight” will be turned on – kinda cool to be able to turn gravity on and off at will!) and the voltage read as a function of time. By analyzing the $V(t)$ graph on the scope, we will determine the inductance and capacitance in the circuit (the resistance is easy to determine, right?).

**Task 1)** Put the capacitor voltage on the oscilloscope

Wire a battery (you can actually use one of the small dry cells for this one), an inductor and a
capacitor in series. (There will automatically be a resistance, right?). Set the oscilloscope to read the voltage across the capacitor into Channel 1. Channel 1 should be set to 0.5 V per division. The time trace should be set around 1 sec or less per division. With the capacitor disconnected, the voltage will be zero. Adjust the channel 1 signal until it is near the bottom. Then connect the capacitor to the battery. The Channel 1 voltage will jump to about 1.5 volt (the voltage of the battery). This means that the battery has completely charged the capacitor and there is no voltage across either the resistor (no current) or the inductor (no CHANGE in current). (ASIDE: For the spring system, this is like releasing a mass on a spring and then seeing some time later that it is sitting still, but with the string stretched out). If you look carefully, you can see some weird spikes at the place where the voltage jumps. That’s what we want to see (the oscillations BEFORE the “mass” stops). Unfortunately, they happen to fast for us to just take a look. So we’ll have to set the scope to automatically trigger when we connect the battery and to STORE the trace so we can analyze it at our leisure.

Task 2) Set the triggering for the oscilloscope
Press the button labeled “trigger menu”. There will be a set of labels on the right side of the screen, corresponding to the buttons to the right of the screen. These pertain to the type of triggering and stored graph. Near the bottom is a button that cycles through Auto, Normal and Single. Set this to Normal so you can get things set up, but Single when you want to take a “snapshot”. The voltage will be zero until we connect the battery, when it will increase. So we want to trigger on a “rising” “edge”. So set those two buttons to “edge” and “rising”. We want it to trigger immediately after we connect the battery, so set the “trigger level” to just above the level of the voltage with the battery disconnected. Now set the mode to “single”. Hit the “Run/Stop” button until you see “Ready” near the top of the screen. Connect the battery. A trace that looks like a dying oscillation will appear on the screen (if it doesn’t, sing out).

Task 3) Analyze the trace.
The people who study such things tell me that the voltage across the capacitor should satisfy the equation:

\[ V(t) = V \left[ 1 - e^{-\frac{t}{\tau}} \cos(2\pi ft) \right] \]  \hspace{1cm} (6)

The most painstaking analysis consists of digitizing the “snapshot” on the scope. But a simpler method is to measure the maximum and minimum voltages (the peaks and valleys of the voltage) and the time at which they occur. This can be done by simply reading the numbers from the graph, but there is a Cursor button that will make this job a little easier. It is then straightforward to calculate the time constant, \( \tau \), and frequency, \( f \), from Equation (6). They are given by \( \tau = 2L/R \) and \( 2\pi f = 1/\sqrt{LC} \). Measuring R, you can calculate L and C.

Task 3) Repeat
Repeat Task 2 with a different capacitor in the circuit. Can you say anything about the new capacitor just by looking at the new trace?
**Conclusion**

Report coherently on the results of the lab. Among the things you should address are: Confirm or deny the predicted form for the voltage on the capacitor. Determine the inductance of the circuit. Discuss the effect of changing the capacitor. All those things I didn’t think off. Plug the hole in the ozone.