The continuum approximation in electrostatics

In this assignment, you will assess the continuum approximation that we make in order to use calculus to calculate the field (and later potential) of an extended charge distribution. For example, we calculated the electric field a distance \( x \) from a finite line segment of charge (oriented along the \( y \) axis) as:

\[
\vec{E} = \frac{2k \lambda}{x} \frac{L/2}{\sqrt{x^2 + (L/2)^2}} \hat{x}
\]  

(1)

(See your textbook). But we know that charges are NOT continuous, but are, in fact, discrete. Let's compare a line of discrete charges to this continuum approximation (Eq. 1)

Start by placing point charges along the \( y \)-axis, equally spaced. Let the charge on each one be \( q \) and the spacing be the distance, \( a \). That is, place a charge \( q \) at the origin; two more at \( \pm a \), two more at \( \pm 2a \), two more at \( \pm 3a \), and so on, until you have placed a total of \( N \) charges. (Hopefully, it's obvious that \( N \) is an odd number).

**QUESTION 1:** Where are the last two charges located? (Ans: \( \pm (\frac{N-1}{2})a \) )

Now, we calculate the electric field of all these charges at the point \((x,0)\). That is, a fixed point on the \( x \) axis. First, it's easy (I think) to see and almost as easy to prove that the vertical (\( y \)) components of these fields will cancel in pairs (in the same way that the \( y \) component of a line of charge cancels to give Eq. 1). Thus, we'll calculate only the \( x \)- component.

**QUESTION 2:** Calculate the \( x \)-comp of the field (at \((x,0)\)) created by the charge at location \((0,na)\).

(Ans: \( E_{x,1} = \frac{kq}{x^2 + (na)^2} \frac{x}{\sqrt{x^2 + (na)^2}} = \frac{kq}{x^2} \frac{1}{\left[1 + (na/x)^2\right]^{3/2}} \) )

**QUESTION 3:** Calculate the total field and put it in a normalized form:

\[
\frac{E_x}{k \frac{(Nq)}{x^2}} = \frac{1}{N} \sum_{i=1}^{N-1} \frac{1}{\left[1 + (na/x)^2\right]^{3/2}}
\]

(2)

(As I hope you will see, normalizing this expression does two things: enables straightforward (i.e., apples-to-apples) comparison of Eq. 1 to Eq. 2, and also keeps the right side of Eq. 2 from getting either too big or too small. Normalize Eq. 1 in a similar way – then you'll see why I stuck that \( N \) on the left side of Eq. 2.)

Now, the complicated part. The sum on the right is not one that I know how to sum in closed form. If you figure a way, definitely show me. But I suspect you will have to do so numerically. It's possible to do this with a spreadsheet, although I'm not convinced it's the best way. You may use any method you know how to use. Calculate the ratio of Eq. 1 to Eq. 2 for the following parameters: \( N = 1, 10, 30, 100 \) and \( x = a, 10a, 100a \) (so a total of 12 results). How well does the continuum approximation work?