Constant Force: Projectile Motion

Abstract
In this lab, you will launch an object with a specific initial velocity (magnitude and direction) and determine the angle at which the range is a maximum. Other tasks, such as determining the initial velocity, comparing the measurements to model calculations and calculating the angle at which to launch to strike a wall at the highest point may be performed.

Introduction
We have done several labs in which a constant force was applied to an object, resulting in a constant acceleration (as indicated by Newton's Second Law of motion – AKA, the energy principle):

\[
\vec{F}_{NET} = m \vec{a} \quad (1)
\]

(Recall that since \( m \) is a scalar, the acceleration is in the same direction as the new force).

In those experiments, we simplified matters by starting at zero initial velocity. Then, the velocity was always in the same direction as the acceleration (and so the same direction as the force):

\[
\vec{v}_f = \vec{a} t \quad (2)
\]

Again, a scalar multiplying a vector gives a new vector in the same direction. But this will not be true for the case we're doing today, where the initial velocity is NOT zero. Although the CHANGE in velocity will be in the same direction as the acceleration:

\[
\vec{\Delta v} = \vec{v}_f - \vec{v}_0 = \vec{a} t \quad (3)
\]

the FINAL velocity will NOT be in the same direction as the acceleration:

\[
\vec{v}_f = \vec{v}_0 + \vec{a} t \quad (4)
\]

Note that the final velocity is the sum of two vectors (the initial velocity and the change in the velocity), and will NOT in general be in the same direction as either of those two. The figure shows the parabolic path of a projectile, with the velocity shown at two points. At the side is shown the vector addition given in Equation (4):

\[
\vec{v}_f = \vec{v}_0 + \vec{a} t
\]

Note that all three vectors (\( \vec{v}_0, \vec{v}_f, \vec{a} t \)) are in different directions. (You should draw a velocity vector with correct direction AND magnitude at the small cross. Draw the analogous vector addition diagram. Note well that there's no guarantee that the triangle you

Drawing 1: Parabolic projectile path with velocity at two points.
make will be a right triangle like mine is.)

“Neglect Air Drag”

When considering projectiles, you’re told invariably to “neglect air drag”. So much that it almost seems like physicists are a little defensive about air drag. We aren’t. It just really is the case that most of the time (not ALL of the time) air drag is a small effect compared to weight. You can estimate the likely effect of fluid drag (notice how I changed “air” to “fluid”? Tricky.) in the following way.

Using his Second and Third Laws of Motion, Isaac Newton developed an expression for the magnitude of the force exerted by a moving fluid upon an object:

\[ |\vec{F}_{\text{drag}}| = \frac{1}{2} \rho A v^2 \]  

(You should check that the units work OK.) To be fair, this expression was already known. Leonardo DaVinci had already argued that drag forces were proportional to cross sectional area (A) and Galileo had claimed that drag was proportional to the density of the fluid (\( \rho \)). And a scientist named Mariotte had supposedly shown that the drag force was proportional to the square of the speed (but I haven’t been able to track this down for sure). But Newton deserves some credit for showing that this expression follows from the fundamental laws of motion (on the other hand, it’s always easier to get the right answer if you already know what the right answer is :). A couple of hundred years later, in the early 20th century, the development of fluid dynamics was used to show that the equation for drag looks like:

\[ |\vec{F}_{\text{drag}}| = \frac{1}{2} C_D \rho A v^2 \]  

See the difference? It’s not easy. The only difference is that there is a drag coefficient, \( C_D \). And the drag coefficient only varies from about 0.1 (for something very streamlined) to about 1 (for something blocky, like a truck). So the basic physics is in Equation (5).

To understand something about this expression, imagine that I release an object from high up. Initially, the only force acting will the weight, acting down. So the object will begin to accelerate downward. As the speed increases, the drag force, acting opposite the motion (or up) will increase. So there will be two forces, the larger one (weight) acting down and the weaker one (drag) acting up. So the net force will be smaller. I can write Newton’s Second Law of motion as:

\[ \vec{F}_{\text{NET}} = \vec{W} + \vec{F}_{\text{drag}} = W \hat{d}own + \frac{1}{2} C_D \rho A v^2 \hat{u} = m \hat{a} \]  

Now, noting that up and down are opposite directions (that is, \( \hat{d}own = -\hat{u} \)), I can rewrite (7) as:

\[ (W - \frac{1}{2} C_D \rho A v^2) \hat{d}own = m \hat{a} \]  

So we see that as long as the weight is greater than the drag (in this case, it will be), the acceleration will always be down. That is, the object will continually speed up. But as the speed increases, the NET force will get smaller and smaller, so the acceleration (while still pointing down) will get smaller and smaller. Eventually, (if I drop the object from high enough), the acceleration will get so small that
it might as well be zero (i.e., too small to measure). And that means that the speed will stop increasing. The object won't just STOP. Zero acceleration doesn't mean mean zero velocity. Zero acceleration just means UNCHANGING velocity. So the velocity of a falling object will approach a constant magnitude obtained from setting the force in Eq. (8) to zero:

\[ v_{\text{TERM}} = \sqrt{\frac{2W}{C_d \rho A}} \]  

(9)

This gives a way of calculating the terminal velocity of various objects. Shortly, we'll see how this will allow us to assess whether we have to worry about air drag for a projectile. As an example, I'll calculate the terminal velocity of a skydiver. The weight will be in the ballpark of 700 N. The density of air at room temperature is about 1.2 kg/m\(^3\) (you can get this from the ideal gas law). The area of a person falling face first is about \( A \approx (2 \text{ m})(0.3 \text{ m}) = 0.6 \text{ m}^2 \). When in doubt, I use a value of 0.5 for the drag coefficient. Putting it all together, I get a terminal velocity of 60 m/s (about 140 mph, which is what the Interwebs says it is...). But the real goal is to be able to estimate in advance whether air drag will be an issue or not. To do that, go back to Equation (6). I can use the expression for terminal velocity (Eq. (9)) to rewrite the magnitude of the drag force as:

\[ |\vec{F}_{\text{DRAG}}| = W \left( \frac{v}{v_{\text{TERM}}} \right)^2 \]  

(10)

There's nothing really new here (this is really the same as Eq. (6)) but it's in a form that tells us something important. The drag force will be small, COMPARED TO THE WEIGHT, whenever the speed is less than the terminal velocity. And it goes as the square, so that if the speed is one-tenth of the terminal velocity, the drag force will be one percent of the weight (right?). So we can now see a way to tell pretty easily whether we have to worry about a drag force or not. Estimate the terminal velocity (from Eq. (9)) and use it in Eq. (10). For example, when Usain Bolt set the world record in the 100 m race, was air drag a significant factor? Using the results (100m in 9.58 seconds), the drag force is about:

\[ |\vec{F}_{\text{DRAG}}| = W \left( \frac{10.44 \text{ m/s}}{60 \text{ m/s}} \right)^2 = 0.03W \]  

(11)

So, no, air drag is not a significant factor, at least compared to his weight. (On the other hand, 3% can be a big deal in world records – consider that the record before Usain Bolt was 9.74 seconds, by Asafa Powell). In today's lab, you'll launch a sphere made of steel or wood or hard plastic. Should you be worried about air drag in interpreting your results?

**Constant Acceleration**

If we DO ignore air drag, then the kinematics is easier. The acceleration is constant:

\[ \ddot{a} = \ddot{g} = 9.8 \frac{\text{m/s}^2}{\text{s}} \text{ down} = 9.8 \frac{\text{m/s}^2}{\text{s}} (-\hat{u}p) = -9.8 \frac{\text{m/s}^2}{\text{s}} \hat{u}p \]  

(12)

so it makes sense to orient one axis of our two-dimensional problem along that acceleration. Conventionally, people orient the y-axis along \( \ddot{g} \) (but with the positive direction up instead of down – weird). So that's what we'll do. Then, the kinematics is easy. The generic expression for position
when the acceleration is constant:

\[ r = r_0 + v_0 t + \frac{1}{2} a t^2 \]  

(13)

becomes two equations (one for x and one for y):

\[ y = y_0 + v_{oy} t - \frac{1}{2} g t^2 \]  
\[ x = x_0 + v_{ox} t \]  

(14)

I want to find out how far the object will fly until it hits the ground ( \( y = 0 \) ), so I call the horizontal distance travelled the range, \( R = x - x_0 \), and the initial height, \( y_0 = H \). After a whole bunch of algebra (I lay it out for you later), the equation for the range becomes:

\[ R = \frac{v_0^2}{g} \sin \theta \cos \theta \left[ 1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2 \theta}} \right] \]  

(15)

where \( v_0 \) is the initial speed and \( \theta \) is the angle at which it is launched. There’s no easy way to use this formula to find the maximum range, but it is straightforward to find it experimentally. Just load up and fire. And use the equation to compare the measured range to the calculated range. But note that you need everything on the right side. So see if you can figure out how to measure each of those parameters.

**The Experiment**

In short, the experiment consists of launching a projectile at varying angles (but keeping as many other parameters the same as possible) to find the maximum range and the angle at which it occurs. Note that you will need to figure out how to find the initial speed and initial height. You may use \( g = 9.8 \text{N/kg} \). Measuring the distance to the landing point is easier if you lay a piece of paper down with a piece of carbon paper on top – the projectile will make a mark on the paper that you can easily see.

**Conclusions**

As you may have guessed, the maximum range will NOT occur at a launch angle of \( 45^\circ \). Why do you suppose that is? Your lecture teacher didn’t lie, so apparently there is some difference between what we’re doing here and what your lecture teacher showed you. Do you suppose you could figure out how to launch a projectile at a wall a fixed distance away so that it hit the wall at the highest point? If your projectile hits the highest, you win.
APPENDIX

To see how the range equation comes about, start with the kinematic equations for 2D motion:

\[ y = y_0 + v_{0y} t - \frac{1}{2} g t^2 \]
\[ x = x_0 + v_{0x} t \]  

(16)

Make the same definitions as before: the landing point is the ground ( \( y = 0 \) ), the horizontal distance travelled the range, \( R = x - x_0 \), and the initial height, \( y_0 = H \). Then the kinematic equations become:

\[ 0 = H + (v_0 \sin \theta) t - \frac{1}{2} g t^2 \]
\[ R = (v_0 \cos \theta) t \]  

(17)

(Note that I’ve used the initial velocity, \( \vec{v}_0 = v_{0x}\hat{i} + v_{0y}\hat{j} = v_0(\cos \theta \hat{i} + \sin \theta \hat{j}) \)). I can solve the x-equation for time ( \( t = R/(v_0 \cos \theta) \) ), substitute this into the equation for y, and get an equation that tells me how far the object goes (but not when it gets there):

\[ 0 = H + \tan \theta \cdot R - \frac{g}{2 v_0^2 \cos^2 \theta} \cdot R^2 \]  

(18)

So far, the algebra is straightforward and well within your capabilities. But the next step is kind of tricky (it involves the quadratic formula), so THIS you aren’t responsible for. If you DO go through it, you get the final expression for the range:

\[ R = \frac{v_0^2}{g} \sin \theta \cos \theta \left[ 1 + \sqrt{1 + \frac{2gH}{v_0^2 \sin^2 \theta}} \right] \]  

(19)

This is Equation (15). Note that the maximum range will not be when the launch angle is 45 degrees. As the figure shows, an angle that gives smaller range AT THE SAME HEIGHT will give a longer range if the landing point is below the launch point. (Essentially, the lower height gives more time for the larger x-velocity to give a larger x-distance).