Constant Force – Free Fall

Abstract

In this lab, you will measure the time for various objects to move a sequence of distances, under the influence of gravity, to determine:
A) Do the various objects move with constant acceleration? and
B) If so, is the acceleration the same for the various objects?

Introduction

Roughly 2300 years ago, Aristotle made the perfectly sensible argument that a small, light object (being pulled downward with less force – hence, “light”) would fall more slowly than a large, heavy object (being “heavy” because it was being pulled downward with greater force). It seemed logical enough that it took 1900 years before anybody got around to checking. That someone was Galileo Galilei, who proved empirically that Aristotle’s statement was false (incidentally inventing kinematics and the scientific method along the way). (By the way, in proving that his professors’ beliefs were completely wrong, he managed to lose his job). Let's see what all is involved in this controversy.

The raison d'être of this class is communicate fully the meaning of Newton's Laws of motion:

\[ \vec{F}_{NET} = \frac{d \vec{p}}{dt} = m \vec{a} \] (1)

(Newton's 2nd Law, aka, the energy principle or momentum update formula)

and

\[ \vec{F}_{12} = -\vec{F}_{21} \] (2)

(Newton's 3rd Law, aka, the momentum principle)

Not only is this enough for an entire semester, you can actually spend years trying to understand the intricacies of those two simple rules. So we'll start small: a constant force.

First, the SIMPLEST case is a force that's constant at zero. Then, from equation (1), we know that the acceleration is zero. From the definition of acceleration:

\[ \vec{a} \equiv \frac{d \vec{v}}{dt} \] (3)

an acceleration of zero means a velocity that doesn't change (i.e., a constant velocity). Note that velocity is a vector and so has a magnitude and direction. So for the velocity to be constant, BOTH the magnitude AND the direction must remain the same. The object must travel at the same speed AND in the same direction (at constant speed in a straight line). (This is Newton's First Law – we see that his FIRST law is really just a special case of his Second Law). Unfortunately, it's terribly difficult to do a physics lab with zero force, as instructive as that might be. (You can SIMULATE a zero force environment, but only if you have a space station or $5000 per student to spend (https://www.gozerog.com/) – we have neither). So we'll move to the second easiest case: a constant (not-zero) force. Again, as force is a vector, a constant force has a magnitude AND direction that cannot change. As it turns out, an experiment with constant force is pretty easy to set up. We'll do that in today's lab.

The Lab

As before, the mechanics of the lab are straightforward. We will use a ramp and stopwatch to measure the time for a cart to roll a set distance. (It would be nice to simply drop the objects, but we
don't have the technology for that – or at least not enough for everybody). Our experiment is very much like Galileo's experiments. Without electronic or mechanical stopwatches, he was reduced to using his pulse for time measurement. You can use your phone (even my old LG Env3 had a decent stopwatch in it) or I'll loan you a cheap plastic stopwatch (that keeps pretty good time). A ramp will be placed a slight angle (anything more than about $2^\circ$ is too much). A frictionless (ok, a VERY low friction) cart will be placed on the ramp and the time to roll a specified distance will be measured and recorded. You should repeat the measurement many times (and remember my answer to “how many?”). You may assume that the error in the time is random (but it would be so choice if you proved it), so the best estimate for the time is the average and the estimate of the error will be the standard deviation (see section 4.2 of the lab textbook). You should do this for a variety of distances (I'll explain why in the Analysis section). The shortest practical distance will be about 0.1 m and the largest possible is about 2 m.

**Analysis**

To determine whether the acceleration is constant, we need to know what the data SHOULD look like if the acceleration IS constant. Most physicists would say that you have to know how to solve differential equations to do this. And, in time, you WILL be required to know how to do that. But we can actually go kind of far without it (and maybe learn some physics to go with all the math you will eventually learn). One thing that will be helpful is to remember that the definition of acceleration:

$$\vec{a} \equiv \frac{\Delta \vec{v}}{\Delta t}$$  \hspace{1cm} (4)

But this is ALSO the slope (rise over run) of a graph of $\vec{v}(t)$ as a function of time. So when a I say that the acceleration is constant, I'm also saying that a plot of $\vec{v}(t)$ as a function of time will have a constant slope (that is, it will be a straight line). And the equation of a line is: $\vec{v}(t) = \vec{v}(0) + \vec{a}t$ – dependent variable ($\vec{v}(t)$) equals a constant ($\vec{v}(0)$, called the intercept) plus another constant ($\vec{a}$, called the slope) times the independent variable ($t$). So in the case that the force is constant, Newton's Second Law says that the acceleration will be constant and from the way we define acceleration, this says the velocity will be linear in the time. We define average velocity in a similar way (as the rate at which displacement changes):

$$v_{AVG} \equiv at$$  \hspace{1cm} (5)

where $\vec{r}$ is the position (basically, the distance from an arbitrarily chosen point). Today's experiment makes our life a little easier than it will be later in the semester – first, the initial velocity is zero (the object starts from rest) and the object falls straight down (we only have to worry about vertical motion for the moment). So the distance the object falls ($D$) becomes:

$$D = v_{AVG}t$$  \hspace{1cm} (6)

and the velocity at the moment it hits the pad can be written:

$$v(t) = at$$

Finally, it's not easy to prove without integral calculus, but I think you'll be willing to accept (at least for now) that the average velocity is just the initial and final velocities added together and divided by 2:

$$v_{AVG} = \frac{v(0) + v(t)}{2}$$  \hspace{1cm} (7)

Remembering that the initial velocity is zero, this can all be put together as:

$$D(t) = \frac{1}{2}at^2$$  \hspace{1cm} (8)

(You should do the algebra leading to equation (8). If it's easy, it won't take long. If it's not easy, you've identified a problem you DEFINITELY need to fix – I’ll help.) That is, the distance travelled increases as the SQUARE of the time. In general, the longer you fall, the farther you move ($D \sim vt$).
but if you're accelerating then you're ALSO speeding up \((v \sim t)\) so that \(D \sim t^2\). This is the equation we'll use to analyze the data and determine whether the acceleration is constant (and \(D\) varies as \(t^2\)) and whether the acceleration of different objects is the same.

**Nonlinear relationships**

The simplest way to determine the acceleration is to simply solve equation (8) for \(a\). The problem with that solution is that it ASSUMES that the acceleration is constant (the question we're trying to answer). So the FIRST job is to prove that Equation (8) really does describe what's going on. First, remember that if the average velocity is constant, then Eq. (6) predicts that the distance travelled will increase linearly with time, but that if the average velocity increases with time, then the distance increases faster than linear (and if the average velocity decreases, then the distance increases more slowly than linearly). So, if we were to plot the distance against time for these three cases, it might look like:

Make sure you can figure out which is which (speeding up, etc.). The problem for us is that we EXPECT the speed to increase, but we're interested in whether it increases UNIFORMLY (constant acceleration). Which of the following curves shows constant acceleration?

It's hard to say. They are BOTH speeding up (right?) but it's not at all obvious which is speeding up UNIFORMLY. But we have an advantage. In our case, the initial velocity is zero. So Equation (8)
tells us that the distance \textbf{MUST} vary as the time squared. So if I plot the same data as above, but use the time \textbf{SQUARED} as the independent variable, then the graph looks like:

\begin{center}
\includegraphics[width=0.5\textwidth]{distance_vs_time_squared.png}
\end{center}

In this case, it's pretty clear which object is \textbf{accelerating UNIFORMLY}, because the SLOPE of the \textit{D vs \textit{t}²} graph is constant (the graph is linear). I've laid this out pretty clearly because it's the first time. We will do a lot of this sort of thing and you'll have to do it on your own (especially on the test) so go over this easy example carefully so that you understand the idea.

\section*{Error in the SQUARE of time}

How to get the error in the time measured is easy -- your text shows how to do it on p. 99 in Equation 4.6. So you'll measure the time to drop a certain distance multiple times (how many?) and calculate the average as the best estimate of the time, and the standard deviation as the best estimate of the error in the measurement (and the Standard Deviation of the Mean -- p. 102 of your lab textbook -- as the best estimate of how far your average is from the "correct" value).

But we need the error in the SQUARE of the time (σ\textit{t}²). The natural thing to do is to simply square the error in time (σ\textit{t}²), but we'll use a simple example to see that this is a dramatic underestimate of the correct result. Consider a real square -- if we want to find the area, we just square the length of one side: \(X²\). But if there is some error (σ\textit{X}) in our measurement of the side, there will be an error in the value of the area, that we can find by calculating:

\begin{equation}
\text{Difference in area} = \text{area WITH error} - \text{area WITHOUT error} = (X + \sigma_X)^2 - X^2 = 2X\sigma_X + (\sigma_X)^2 \approx 2X\sigma_X
\end{equation}

So we see that the error in the area of the square of the time can be calculated as twice the average time, multiplied by the error in time and \textbf{NOT} as the square of the error in time (that is the tiny yellow corner that we're ignoring as being very small). We'll develop a more systematic way to do this using derivatives later (Chapter 3 in your lab text).

\section*{Conclusions}

At the end of it all, you should have a graph for at least two different objects (three is better; five is awesome) showing the distance they fell as a function of the square of the time it took them to fall that distance. This representation of the data should allow you to answer the questions raised in
today's lab. (Remember what they were?) Finally, if the acceleration IS constant for any of the objects, you will want to know if they are the same. Suppose that you find that the acceleration of one is 
\[(0.55\pm0.02) \text{ (m/s)/s}\] and another is \[(0.47\pm0.08) \text{ (m/s)/s}\]. Are the two accelerations the same? How about if one is \[(0.202\pm0.005) \text{ (m/s)/s}\] and another is \[(0.210\pm0.007) \text{ (m/s)/s}\]?

Whatever your result, the question of whether all objects respond to gravity in exactly the same way is very significant. At first, Aristotle's idea that an object twice as heavy will fall a given distance in half the time seems reasonable. When you do the experiment, you find that he's wrong (at least about the degree of difference). But the idea that all objects respond IDENTICALLY to gravity is also strange. Consider:

According to Newton's Second Law of Motion (the energy principle), an object subject only to a gravitational force exerted by the Earth is given by:

\[
\vec{a} = \frac{\vec{F}_{\text{GRAV}}}{m_{\text{INERTIA}}} \quad \text{(9)}
\]

where \(m_{\text{INERTIA}}\) is the inertial mass, the property of matter that makes it harder or easier to accelerate (an empty grocery cart is easier to accelerate than a full cart because it has less inertial mass when empty). Newton's Law of Universal Gravitation says that the gravitational force the Earth exerts on an object is given by:

\[
\vec{F}_{\text{GRAV}} = m_{\text{GRAV}} \vec{g} \quad \text{(10)}
\]

where \(\vec{g}\) called the gravitational field and depends only on properties of the Earth (radius, mass, etc.). So putting (9) and (10) together, we get:

\[
\vec{a} = \frac{\vec{F}_{\text{GRAV}}}{m_{\text{INERTIA}}} = \frac{m_{\text{GRAV}} \vec{g}}{m_{\text{INERTIA}}} \quad \text{(11)}
\]

So the observation that objects fall to the ground together requires that two properties (gravitational mass and inertial mass) be the same when they really don't have anything to do with each other. A man named Baron Loránd Eötvös de Vásárosnamény (a Hungarian, like my Mom) did experiments to test this. Strictly speaking, he only showed that the two masses are proportional to one another (the ratio doesn't have to be 1; it just has to be the same for all materials). But it became even more important when Einstein developed his theory of General Relativity – THAT theory required that they be IDENTICAL. So a test of the acceleration of a falling object is a test of whether Einstein's theory of General Relativity is correct. Kind of a big deal.