Static Equilibrium (1) – Addition of vectors

Abstract

In this lab, you will repeatedly set up an object at static equilibrium and confirm graphically and computationally that the new force is zero.

Introduction

As we’ve noted, Newton’s second law of motion:

\[ \vec{F}_{\text{NET}} = m\vec{a} \quad (1) \]

(the energy principle or momentum update formula) shows that an object subject to a constant, ZERO force has an acceleration of zero. And from the definition of acceleration, a zero acceleration implies a CONSTANT velocity. This is somewhat counterintuitve. People tend to believe that the natural state of motion is at rest (as stated by Aristotle). But Galileo made clear from his experiments that the natural state of motion is continuous motion:

1) at a constant speed and
2) in the same direction.

(Shorthand for points 1) and 2) above is “constant velocity” - \( \vec{v} = \text{constant} \). The counterintuitive point is that an object not subject to any force can move – and as far as we know, it will move forever. The only way that object will STOP is if you exert a force upon it. So when a car slows to a stop, it's because of the force of the road or the air pushing against the car. If an airplane is flying at constant velocity, then the force acting upon it is zero. In short, the ONLY reason objects speed up, slow down or turn is if a force acts upon them. But this ALSO seems counterintuitive. We know that a plane flying at constant velocity has forces acting upon it. They are called lift, thrust, drag and weight. (Three of these forces are contact forces exerted BY the air ON the plane – the fourth is a magical force that reaches through empty space. Make sure you can identify which is which). So what Eq. (1) really says is that the acceleration is related to the OVERALL overall force exerted on an object. If all the forces added together add to zero, THEN the acceleration is also zero. But if the forces are not in balance (the OVERALL or NET force is NOT zero), then the object will accelerate or, equivalently, the velocity will change.

When Does 5+5 Equal 5?

(see: http://www.physicsclassroom.com/Class/vectors/u3l1b.cfm)

It would be great if we could add forces just like we add numbers – 5+5=10. But unfortunately for us, it doesn't work that way and it's easy to see why. Suppose you're shopping for groceries and you're pushing forward with a force of 5 lbs and your five year old is pushing BACK with 5 pounds – clearly the cart will not accelerate. But if you and the child are both pushing forward, the cart WILL accelerate, and more than when you push alone. So in one case, we see that 5+5=10, but in the other, 5+5=0. And if I vary the two directions, I can get 5+5=anything between 0 and 10. The reaction of an object to a force depends not only on the MAGNITUDE of the force (how hard you push or pull), but also on the DIRECTION of the force – in short, force is a VECTOR. So if I want to add forces, I have to add vectors.

There are two methods that we will use to add vectors:

1) a graphical method, where we represent a vector as an arrow, whose length represents the magnitude of the vector and whose direction represents, well, the direction; and

2) a computational method, in which we will use trigonometry to convert the addition of vectors into
two (or more) regular, numerical additions.

**Graphical Method**

First, consider the examples above – two 5 pound forces. If they both point in the same direction:

![Graphical Representation](image)

you can see that the combined effect of the two forces is to provide the same result (i.e., the same acceleration) as one 10 pound force. Note also that we've added the vectors by placing the tail (non-pointy end) of one vector at the head (pointy end) of the other, and constructed the sum from the tail of the FIRST vector to the HEAD of the other. Finally, note that the order of the addition is unimportant. I could just as easily put the blue arrow first, and place the tail of the green arrow at the head of the blue arrow. The result would be the same:

![Graphical Representation](image)

All of these properties hold whether I have two arrows or many more, and whether they point in the same direction or not. Now let the two arrows point in opposite directions:

![Graphical Representation](image)

Now we see that the head of the last arrow coincides with the tail of the first – this means the result of the two vectors is a vector whose magnitude is zero – the equivalent of no force (a resulting force of zero – this is an important special case). Another case is if you're pushing forward and the kid is pushing to the right:

![Graphical Representation](image)

In this case, the overall force is neither 10 nor zero, but something in between. If you measure the length of the gray arrow (the resulting force), you get something around 7.1 lbs. (In fact, you get a length of about 3.5 inches – so you need to think a little about force vectors and the pictures I have been showing and how the length of a drawing does or does not relate to the magnitude of a force vector). The angle you measure is -45°. (Why negative – what does the negative mean?) So we've determined the OVERALL force is about 7.1 lbs at a direction of -45°. That is, a 5 lb force acting in the direction 0° and another 5 lb force acting at -90° is completely equivalent to a 7.1 lb force acting in the direction -45°. So now if a bystander pushes the cart with a force of 7.1 lb in the direction -45° + 180° = 135°, the picture looks like:
That is I can balance the two forces with a single force that balances the RESULT of the two forces. Your task will be to graphically (and then computationally) show that three or more forces that you KNOW to be balanced add up to zero. (How do you suppose you can tell if all the forces on an object add up to zero?)

**Computational Method:**

Another way to add vectors is to agree to write all vectors as a set of components parallel to a set of agreed upon axes. That is, I can decide that the direction the cart is pointed (the 0° direction) is the positive x axis and the direction to the left (the 90° direction) is the positive y axis. This is arbitrary and you can make other choices, but this is a conventional choice. Then the first force above (5 lbs at 0°) can be written in two ways:

1) \( \vec{F}_1 = (5 \hat{i} + 0 \hat{j} + 0 \hat{k}) \text{ lbs} \) This is the way that me and your Tipler text like to do it. The \( \hat{i}, \hat{j} \), and \( \hat{k} \) are how you tell which part points along the x, y and z axes, respectively. Note that I can take the units outside the parentheses, because both components have the same units – I’m effectively “factoring out” the units, just like in algebra.

2) \( \vec{F}_1 = \langle 5, 0, 0 \rangle \text{ lbs} \) This is the way that Dr. Allain, your Chabay and Sherwood text, and mathematicians like to do it. This is shorter, but you have to agree that you will ALWAYS write the units in the specified order (x, then y, then z). In the first notation, you can change the order of the components and it’s still correct (pointless, but correct). In the second notation, if you change the order, you completely change the vector. (In what follows, I’ll ignore the z-component because it will always be zero, at least for today’s lab).

The second vector would then be written:

\[ \vec{F}_2 = (0 \hat{i} + (-5) \hat{j} + 0 \hat{k}) \text{ lbs} \] or \( \vec{F}_2 = \langle 0, -5, 0 \rangle \text{ lbs} \). To add them, I simply add the corresponding numbers as we always add numbers. So the \( 5 \hat{i} \) in \( \vec{F}_1 \) combines with the \( 0 \hat{i} \) in \( \vec{F}_2 \) to give a result of \( 5 \hat{i} + 0 \hat{i} = (5+0) \hat{i} = 5 \hat{i} \). Similarly, we combine the y-parts: \( 0 \hat{j} + (-5) \hat{j} = (0-5) \hat{j} = -5 \hat{j} \). That is, \( \vec{F}_1 + \vec{F}_2 = [(5+0) \hat{i} + (0-5) \hat{j}] \text{ lbs} \) = \( 5 \hat{i} - 5 \hat{j} \) \text{ lbs} \). (Using the other notation, this is: \( \vec{F}_1 + \vec{F}_2 = \langle 5, 0, 0 \rangle \text{ lbs} + \langle 0, -5, 0 \rangle \text{ lbs} \) = \( \langle 5, -5, 0 \rangle \text{ lbs} \). Either way, I would now like to know result in the conventional format – a magnitude and a direction. This is easy once you realize that the two parts of the vector ( \( 5 \hat{i} \) and \( -5 \hat{j} \) ) are two sides of a right triangle. So the magnitude can be calculated using the Pythagorean Theorem and the direction using the definition of tangent. You find that the magnitude is: \( |\vec{F}_1 + \vec{F}_2| = \sqrt{(5 \text{ lbs})^2 + (5 \text{ lbs})^2} = \sqrt{50 \text{ lbs}^2} = 7.1 \text{ lbs} \) and the direction is \( \theta = \tan^{-1}[-5 \text{ lbs} / 5 \text{ lbs}] = -45^\circ \), both consistent with the graphical solution. (ASIDE: Do the same for the vector: \( \vec{F} = (5 \hat{i} + 5 \hat{j}) \text{ lbs} \)
\[-5,5,0 \text{ lbs} \text{ – that is, using both graphical and computational methods, find the magnitude and direction of this vector). We’ll use both of these methods to add at least three vectors that are known to add to zero. Any difference we see will be due to some error somewhere – a part of today’s lab will be to try to identify the source of that error.\]

**The Lab**

You will suspend a weight from a string and use two spring scales to balance the weight. Knowing that the knot at the center is at equilibrium, show graphically and computationally that the net force is zero. You will have to set up a coordinate system. Your instinct will be to set up the x axis horizontally and the y axis perpendicular so that y is positive up. This will definitely work. But it may or may not be the most convenient way.

**Conclusions**

Since you know that the knot is at equilibrium, any deviation from zero for the net force must be due to some error somewhere. Things to consider are: what is the precision with which you can measure the forces and angles? Is the error within that precision? Are there forces acting that we don’t consider? If so, estimate the magnitude of those forces to argue whether they are possible explanations. For example, the Sun exerts a force upon the hanging weight. We can estimate the magnitude of that force from:

\[
\frac{F_{\text{SUN}}}{F_{\text{EARTH}}} = \frac{G M_{\text{SUN}} m}{GM_{\text{EARTH}} m} \frac{r_{\text{SUN}}^2}{R_{\text{EARTH}}^2} \approx 10^{-3}
\]

So, if the error you see is about 0.1 % of the weight of the object, the force of attraction from the Sun is a POSSIBLE explanation. What would you have to do to PROVE this explanation?

Imagine that you used a LOOP of string, like this:

What is the minimum number of measurements you have to make to determine the weight of the hanging object?