Conservation of energy and collisions

Abstract

In this lab, you will roll a cart down a ramp tilted at an angle $\theta$, observe it rebound from the wall, and quantify the conservation of mechanical energy. It is more than likely that you will observe that mechanical energy is not conserved, and so the “coefficient of restitution” will be introduced. Relevant questions are whether this coefficient is independent of the speed of the collision (as is often claimed without proof) and if so, what is its value (including the error).

Introduction

I have claimed that the raison d'etre of this class is to understand and use Newton's Equations:

$$\begin{align*}
    a) \quad F_{NET} &= \frac{d \vec{p}}{dt} = m \vec{a} \\
    b) \quad \vec{F}_{12} &= -\vec{F}_{21}
\end{align*}$$

(1)

To this end, we have learned about free body diagrams, various forces that add as vectors to give the net force, and kinematics (i.e., position, velocity, acceleration, and how they are related). (I've also tried hard to keep you from confusing Newton's Second Equation – a) above – with Newton's Third Equation – b) above). And if you are interested in things like the final direction of an object or the time for some event to occur or, for a specific example, the distance an object travels after rolling off a table, this is the only game in town – find the forces and solve the kinematic equations. But sometimes, you might be interested in things NOT having to do with the vector nature of these equations, or the time for them to occur. In such cases, physicists have invented a mathematical trick to short circuit a lot of this work. It's called “energy” (and the savings can be considerable – finding the empirical efficiency of your car's internal combustion engine is fairly easy using energy and virtually impossible using Newton's equations from above). Inventing energy starts with Newton's Second Equation above [I'll do everything in one dimension; know that it all works OK in three dimensions, just with more wailing and gnashing of teeth]:

$$F_{NET} = \frac{\Delta v}{\Delta t} = m \frac{\Delta v}{\Delta t} \frac{\Delta x}{\Delta t} = \frac{\Delta v}{\Delta x} \frac{\Delta x}{\Delta t} = m \frac{\Delta v}{\Delta x}$$

(2)

(Please note carefully each step above – they're not trivial but not overly complicated). Finally I can rearrange this ever so slightly into:

$$F_{NET} \Delta x = m \Delta v$$

(3)

If you go back to the earlier lab in which I showed how to find the error in the square of time, and change the “t” to a “v”, you can see how we can write:

$$v \Delta v = \frac{1}{2} \Delta v^2$$

(4)

Finally, if we note that the mass is a constant (and so I can either find the difference in $v^2$ and then
multiply by \( m - m \Delta v^2 \) – or multiply \( m \) by \( v^2 \) and then find the difference – \( \Delta mv^2 \), we can finally get:

\[
F_{\text{NET}} \Delta x = \Delta \left( \frac{1}{2} m v^2 \right)
\]

(5)

The left side is called the “work”, which kind of makes sense – if you push or pull an object with a force, \( F_{\text{NET}} \), through a distance, \( \Delta x \), you may well feel like you have done work. The left side is called the “kinetic energy” and is energy associated with motion (since it depends on the speed). Note that there is no direction here – even if you reverse the direction of the object, the speed is squared and so the minus sign vanishes (and I'll ask you to trust me that even if we do the hard work of doing this in three dimensions, the vector signs STILL disappear – awesome). There is a hint of direction, though, on the left side. “Work”, as we've defined it, can be positive or negative or even zero. Consider the case that the displacement, \( \Delta x \), and the force, \( F_{\text{NET}} \), are in the same direction. From the kinematics you studied, you know that if a force pushes you in the direction you are travelling, you “speed up” (this is what the “accelerator pedal” in your car does – it causes the road to push you in the direction you are already travelling). And note that this is what Eq. 5) shows – if \( \Delta x \) and \( F_{\text{NET}} \) have the same sign, then the product is positive and the change in kinetic energy is positive (that is, the kinetic energy increases which it must if the speed increases). On the other hand, if the force and displacement are in opposite directions, kinematics shows that you slow down (this is what the brake pedal does – it makes the road push opposite the direction you're travelling and so you slow down). And again, this is what Eq. 5) shows – be prepared to argue why kinematics and energy give the same answer in this case (just as I did above). Kinematics also considered the case that the force and displacement are perpendicular. In that case, the speed does not change and the velocity only changes because of a change in direction (this is what the final accelerator in your car – the steering wheel – does; it changes your direction). In 2 or 3 dimensions, you can always break the force vector into a part along the direction of motion (that will change the speed) and a part that will change direction. Only the part along the direction of motion will do work and change kinetic energy.

“it surrounds us; it binds the galaxy together”

Most people (including many physicists who should know better) have a semi-mystical attitude about energy. Obi-Wan Kenobi's quote about the “force” sounds a little like the way most people view energy. But I think it's clear from the above that “energy” is just the name we give to a particular mathematical trick that helps us solve Newton's Equations for the motion of objects (including planets and cars and electrons). What pops out of the mathematical gymnastics is “work” – the product of the force and the displacement, taking care to consider the part of the force along the direction of motion – and “kinetic energy” – the “energy” associated with motion (And since we equate the work to the change in kinetic energy, they have to have the same units. So you should be prepared to show that the units of work – force times distance – are the same as the units of kinetic energy – mass times speed squared).

Note also that it's not “THE force” that binds the galaxy together; it's a force – the force we call “gravity”. And gravity is special when it comes to work. There are nice things about the way gravity works that don't apply to friction, for example, or magnetism. Consider that an object moves completely horizontally. Then at every point, the weight (the gravitational force) is at right angles to the motion and so does no work (and then doesn't change the kinetic energy). And if an object moves only up (or down), the work is the weight (\( mg \)) times the change in altitude (\( \Delta y \)). If \( \Delta y \) is down like the weight, the work is positive and the object speeds up, exactly as your intuition tells you.
But it gets even better if the object moves along some direction other than completely vertical or horizontal:

We can write the work done as the displacement (\( |\vec{\Delta r}| \)) times the part of the force in the direction of the motion (\( W \cos \theta \)). But we get the same result if we write the work as the magnitude of the force (\( W \)) times the part of the displacement in the direction of the force (\( \Delta r \cos \theta \)). For the simple motion in the line, there's no reason to pick one over the other. But if the object moves along a curved path, one is definitely easier. Since the weight is always down (by definition) and the displacement is constantly changing, it's easier to find the part of the displacement that is down (at every step), add them up, and get the net change in height, \( \Delta y \). Since gravity is special in this way, we define something called potential energy that increases as an object goes up. So when an object rises in a gravitational field, it gains potential energy and loses kinetic energy (and the reverse when it descends in a gravitational field).

**Conservation of Energy**

If we define the potential energy as:

\[
\Delta U = - \text{Work} = -mg \Delta y \tag{6}
\]

it's straightforward algebra to rearrange Eq. 5) into:

\[
- \Delta U = \Delta \left( \frac{1}{2} mv^2 \right) \tag{7}
\]

And a little bit more to get:

\[
\Delta \left( \frac{1}{2} mv^2 + U \right) = 0 \tag{8}
\]

The stuff in parentheses is called the mechanical energy and if the CHANGE in something is zero, that thing must be constant (i. e., not changing or “conserved”). This result is the first notion that energy is conserved. But if you introduce other forces, like friction, this mechanical energy is NOT conserved and for a time, physicists felt that if mechanical energy is conserved, that's nice, and if it isn't, well so it goes. But when James Prescott Joule showed in 1845 that the heat generated by friction was equivalent to the work done by friction, and with following work showing that in atomic and nuclear physics energy always seems to be conserved, it was widely held that energy is conserved always (and when the virtually undetectable neutrino was detected, on the basis of energy conservation, there was no more doubt – energy is conserved). Despite these successes, it can be hard in an experiment we do in
lab to account for all the energy (like balancing your checkbook – you can be sure that penny is in there somewhere, without necessarily being able to find it :).

**The Experiment**

In this experiment, you will roll a cart equipped with a spring loaded plunger down a ramp tilted at a small angle, $\theta$. Using the distance, $D$, the cart rolls and the angle of tilt, you can calculate the change in potential energy (what else do you need to know?). This potential energy at the start of the trip down can be related to the kinetic energy at the end of the trip down and so, to the speed as the cart's plunger strikes the wall. The cart will rebound from the wall and the speed the cart moves AWAY from the wall is taken to be related to the speed of approach by a constant called the “coefficient of restitution”. That is, if the speed the cart is moving when it first strikes the wall is written as $v$, and the speed the cart moves as it leaves the wall is written as $v'$, then they are related as:

$$v' = e v$$  \hspace{1cm} (9)$$

where $e$ is the coefficient of restitution. (The coefficient of restitution is 1 for a perfectly elastic collision and 0 for a perfectly inelastic collision; collisions are generally not either and $e$ will have a value somewhere between 0 and 1). The cart will now roll back UP the ramp, converting the kinetic energy just after the collision into potential energy. By measuring how far the cart rolls back up the ramp, $D'$, you can find the final potential energy. By relating the energy AFTER the collision to the energy BEFORE, you can write a relationship between the initial potential energy to the final potential energy involving the coefficient of restitution. That is, by following steps of the form:
A) relate the potential energy at the beginning to the kinetic energy as the cart strikes the wall
B) relate the kinetic energy BEFORE the collision to the kinetic energy AFTER the collision using the coefficient of restitution
C) relate the kinetic energy as the cart moves away from the wall to the final potential energy
you should be able to generate a model for the potential energy at the end, $U'$, to the potential energy at the beginning, $U$, of the form:

$$U' = f(e) U$$ \hspace{1cm} (10)$$

where your task is to do the algebra to find the function, $f(e)$.

**Analysis**

You will measure the distance the cart rolls down BEFORE colliding and the distance the cart rolls back up AFTER colliding and use this to calculate the potential energy at the beginning and the end. The question is whether that data confirms the model in Eq. 10. To do so, you will find use for the LINEST function (whether you use Excel, Openoffice or Google Sheet – they all call it the by the same name and the way to use it is only different enough to be annoying).

**Conclusions**

You will likely find that some of the “mechanical energy” has gone missing. If you're like most physicists, you'll use the standard “get out of jail free card”, and say the energy “lost” is converted to heat. Can you use your knowledge from chemistry to estimate quantitatively if your explanation makes sense?