11.1 a) Recall Eqn 4.37 - we'll write the kinetic energy of charge $q_1$ similarly (but classically):

$$\frac{1}{2}m \dot{r}^2 + \frac{k q_1 q_2}{2mr^2} + \frac{k q_1 q_2}{r} = E$$

Drawing the figure analogously to Fig 11.2:

So (like example 11.1) $\theta = \pi - 2\alpha$

The algebra is simplified if $u = \frac{1}{r}$.

$$\dot{r} = \frac{dr}{dt} = \frac{d}{dt}(\frac{1}{u}) = -\frac{u}{u^2}$$

$$\frac{1}{2}m \left(-\frac{u}{u^2}\right)^2 + \frac{k q_1 q_2}{2m} u^2 + (k q_1 q_2) u = E$$

$$\frac{1}{2}m (\ddot{u})^2 + \frac{k q_1 q_2}{2m} u^6 + (k q_1 q_2) u^5 = E u^4$$
We also prefer \( u \) as a function of angle (call it \( \phi \)) rather than time: \( u(t) \rightarrow u(\phi) \)

\[
\frac{du}{dt} = \frac{du}{d\phi} \frac{d\phi}{dt} = u' \phi
\]

\( \phi \) is related to \( L \) as: \( L = mr^2 \dot{\phi} \)

or \( \dot{\phi} = \frac{L}{m} u^2 \)

\[
\frac{1}{2} m \left( u' \frac{L}{m} u^2 \right)^2 + \frac{L^2}{2m} u^6 + (kq_1q_2) u^5 = Eu^4
\]

\[
\frac{1}{2} \left( \frac{L^2}{m} \right) u^2 u^4 = E
\]

\[
\frac{L^2}{2m} u^2 + \frac{L^2}{2m} u^2 + (kq_1q_2) u = E
\]

\[
\frac{L^2}{2m} u^2 + \frac{L^2}{2m} u^2 + (kq_1q_2) u = \frac{2me^2}{L^2}
\]
11.1e-cont] Consider that this is a conservative force (so E is constant) and a central force (so L = ...). Then both are equal to the initial value: \( E = \frac{1}{2} m u^2 \)

\[ L = m u \cdot b \]

Then \[ \frac{2mE}{L^2} = \frac{m^2v^2}{m^2u^2b^2} = \frac{1}{b^2} \]

\[ \frac{2\pi \left( k_q q_2 \right)}{m^2 u^2 b^2} = \frac{k_q q_2 / E}{b^2} \]

The numerator is some kind of distance (right? Can you prove this?)

- call it \( 2p = \frac{k_q q_2}{E} \)

(What is \( p \) - what is the physical significance?)
11.12 - cont | With this, the energy becomes:

\[ u'^2 + u^2 + \left( \frac{2p}{b^2} \right) u = \frac{1}{b^2} \]

(Are the units ok?)

\[ u'^2 = (-1) u^2 - \left( \frac{2p}{b^2} \right) u + \frac{1}{b^2} \]

We will have use for the roots of the LHS, so we'll find them now:

\[ U_+ = \frac{\frac{2p}{b^2} \pm \left[ \left( \frac{2p}{b^2} \right)^2 + \frac{1}{b^2} \right]^{\frac{1}{2}}}{-2} \]

\[ = -\left( \frac{p}{2b^2} \right) \pm \left[ \left( \frac{p}{2b^2} \right)^2 + \frac{1}{b^2} \right]^{\frac{1}{2}} \]

\[ = -\left( \frac{p}{b^2} \right) \pm \left( \frac{p}{b^2} \right) \left[ 1 + \left( \frac{1}{p} \right)^2 \right]^{\frac{1}{2}} \]

Recall that \( u = \frac{1}{r} \) and so \( u' = -\frac{1}{r^2} \frac{dr}{d\theta} \)

So what do the roots above represent?
11.1a (cont') Which of the roots is positive and which is negative?

So we can write the energy as:

\[
\frac{du}{d\phi} = [u-u_+)(u_-u)]^{1/2}
\]

\[
d\phi = \frac{du}{\sqrt{[u-u_+)(u_-u)]}}^{1/2}
\]

We will integrate from the initial position to the point of closest approach:

\[
\int d\phi = \alpha
\]

\[
\int_{u_+}^\infty \frac{du}{\sqrt{[u-u_+)(u_-u)]}}^{1/2}
\]

\[
= -\sin^{-1}\left\{ \frac{-2u+u_-+u_+}{u_+-u_-} \right\}_{0}^{u_+}
\]
\[ \frac{\theta - \cos^4 \theta}{-\sin^{-1} \left\{ \frac{u_- - u_+}{u_+ - u_-} \right\}} \]

\[ + \sin^{-1} \left\{ \frac{u_+ + u_-}{u_+ - u_-} \right\} \]

\[ = -\sin \left\{ -1 \right\} \]

\[ = -\sin \left\{ -1 \right\} \]

\[ \frac{u_+ + u_-}{u_+ - u_-} = \frac{-2 (\frac{c}{b})^2 - (\frac{c}{b})^2 \sqrt{1 + (\frac{c}{b})^2} + (-1) \left( \frac{c}{b} \right)^2 + (\frac{c}{b})^2}{-2 (\frac{c}{b})^2 \sqrt{1 + (\frac{c}{b})^2}} \]

\[ = \frac{-2 (\frac{c}{b})^2}{-2 (\frac{c}{b})^2 \sqrt{1 + (\frac{c}{b})^2}} \]

\[ = \frac{-1}{\sqrt{1 + (\frac{c}{b})^2}} \]
\[ \alpha = -8\sin(-1) + 8\sin^{-1}\left(\frac{-1}{\sqrt{1+\left(\frac{b}{c}\right)^2}}\right) \]

\[ = -(-\frac{\pi}{2}) + 8\sin^{-1}\left(\frac{-1}{\sqrt{1+\left(\frac{b}{c}\right)^2}}\right) \]

So \[ \theta = \pi - 2\alpha \]

\[ = \pi - \left(\pi + 2\sin^{-1}\left(\frac{-1}{\sqrt{1+\left(\frac{b}{c}\right)^2}}\right)\right) \]

\[ = -2\sin^{-1}\left(\frac{-1}{\sqrt{1+\left(\frac{b}{c}\right)^2}}\right) \]

\[ \sin\left(-\frac{\theta}{2}\right) = \frac{-1}{\sqrt{1+\left(\frac{b}{c}\right)^2}} \]

\[ \csc\left(\frac{\theta}{2}\right) = \sqrt{1+\left(\frac{b}{c}\right)^2} \]

\[ \csc^2\left(\frac{\theta}{2}\right) - 1 = \cot^2\left(\frac{\theta}{2}\right) = \left(\frac{\frac{b}{c}}{\sqrt{1+\left(\frac{b}{c}\right)^2}}\right)^2 \]
11.1 a - cont

Finally: \( b = p \cot(\frac{\theta}{2}) \)

... Long way to go ...

b) \( D(\theta) = \frac{b}{\sin \theta} \cdot \frac{db}{d\theta} \)

\[ = \frac{b}{\sin \theta} \cdot p \frac{1}{d\theta} \left( \cot \left( \frac{\theta}{2} \right) \right) \]

\[ = \frac{b}{\sin \theta} \cdot p \left( -1 \right) \frac{1}{2} \csc^2 \left( \frac{\theta}{2} \right) \]

\[ = \frac{b}{\sin \theta} \cdot p \cot \left( \frac{\theta}{2} \right) \cdot \frac{p}{2} \cdot \frac{1}{\sin^2 \left( \frac{\theta}{2} \right)} \]

\[ = \frac{p^2}{2} \cdot \frac{1}{2 \cdot \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right)} \cdot \frac{\cos \left( \frac{\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} \cdot \frac{1}{8 \sin^2 \left( \frac{\theta}{2} \right)} \]

\[ = \frac{p^2}{4} \cdot \frac{1}{\sin^4 \left( \frac{\theta}{2} \right)} \]
11.1c) \[ V = \int \frac{p^2}{4} \cdot \frac{1}{\sin^4(\frac{\theta}{2})} \sin \theta \, d\theta \, d\phi \]

\[ = \frac{p^2}{4} \int_0^{2\pi} d\phi \int_0^{\pi} \int \frac{1}{\sin^4(\frac{\theta}{2})} \sin \theta \, d\theta \]

\[ = \frac{p^2}{4} (2\pi) \int_0^{\pi} \frac{2 \sin(\frac{\theta}{2}) \cos(\frac{\theta}{2})}{\sin^4(\frac{\theta}{2})} d\theta \]

\[ = \frac{(4\pi)p^2}{4} \int_0^{\pi} \sin^{-3}(\frac{\theta}{2}) \cos(\frac{\theta}{2}) \, d\theta \cdot \frac{1}{2} \]

\[ = \frac{(8\pi)p^2}{4} \int_0^{\pi} \sin^{-3}(\frac{\theta}{2}) \, d\left[ \sin^2(\frac{\theta}{2}) \right] \]

\[ = \frac{(8\pi)p^2}{4} \left[ \frac{\sin^{-2}(\frac{\theta}{2})}{-2} \right]_0^{\pi} \]

\[ \Rightarrow +\infty \quad \text{(right?)} \]
In one dimension (1D), free particles are represented by $e^{\pm ikx}$ (where $k > 0$).

For each region of $x$ (with the scattering region near $x = 0$), we have:

- $f(x) = e^{ikx}$, $x < 0$ - Incident
- $f_R e^{-ikx}$, $x < 0$ - Reflected
- $f_T e^{i k x}$, $x > 0$ - Transmitted

So

$$f(x) = A \left\{ e^{ikx} + f_R e^{-ikx} \quad x < 0 \\
+ f_T e^{i k x} \quad x > 0 \right\}$$

where $A$ normalizes the WF.
11.2 (cont) In 2D, there's no easy (or even kinda-hard) way to show that the asymptotic form of a free particle in polar coords is:

\[ f(r) \sim \frac{e^{ikr}}{\sqrt{r}} \]

other than the author's hand-wavy "conservation of probability". Since the area of a cylinder goes as \( \pi r \), the probability must go as \( \frac{1}{r} \), so that \( f(r) \sim \frac{1}{\sqrt{r}} \). I know that it seems kind of crazy to base a physical theory on the conservation of something made up - like probability. But then we embrace the conservation of other made up things - like energy ... 

\[ f(r) \sim e^{ikx} + f(\theta) \frac{e^{ikr}}{\sqrt{r}} \]