FINITE WELL with an adjustable potential:

\[ \begin{align*}
&\text{The particle is in a well of depth } V_0 \\
&\text{with a divider of height (in energy) of } V_0 \\
&\text{and a width (in space) of } d. \\
\end{align*} \]

As \( d \to 0 \), this is a single, finite well of width (in space) of \( a \) and a depth of \( V_0 \).

As \( d \) increases, so that \( d \gg a \), this is two independent wells of width \( \frac{d}{2} \).

So we can compare these to the single finite well to see that it works.
the wave equation is:

$$-\frac{\hbar^2}{2m} \dddot{y}(x) + V(x) \ddot{y}(x) = E \dot{y}(x)$$

$$\dddot{y}(x) = \frac{2m}{\hbar^2}(\nu - E) \dot{y}(x)$$

Define $u = \nu \theta E_1$, $\varepsilon = \frac{E}{E_1}$, $E_1 = \frac{x_0^2}{\hbar^2}$

(Note that $\nu$ is the GS of an infinite well is $\pi^2 E_1$)

so $\dddot{y}(u) = (\nu - \varepsilon) \dot{y}(u)$

Now, putting in the potential:

...
\[ v(x) = \begin{cases} 
+V_0 & |x| < \frac{d}{2} \\
0 & \frac{d}{2} < |x| < \frac{d+a}{2} \\
+V_0 & \frac{d+a}{2} < |x| 
\end{cases} \]

\[ u(x) = \begin{cases} 
+V_0 & |u| < \frac{d}{2} \\
0 & \frac{d}{2} < |u| < \frac{8 + 1}{2} \\
+V_0 & \frac{8 + 1}{2} < |u| 
\end{cases} \]

where \( d = \frac{d}{a} \)

Since \( v(x) \) is an even function, we know we can make the WF either even or odd, that is, we can focus on the even and odd solutions.
So for region I (|u| < \frac{5}{2})

\[ \psi''(u) = (\nu - \xi) \psi(u) \]

I'm interested in bound states, so I assume \( \xi \leq \nu \) or that \( \psi'' \) and \( \psi \) have the same sign. Then the solution is of the form \( \psi \propto e^{\pm ku} \)

\[ \psi''(u) = k^2 \psi(u) = (\nu - \xi) \psi \]

But \( e^{\pm ku} \) are neither even nor odd. **But** I can make them even or odd with a linear combination.

Even: \( \frac{e^{+ku} + e^{-ku}}{2} \) (\( = \cosh(\nu u) \))

Odd: \( \frac{e^{+ku} - e^{-ku}}{2} \) (\( = \sinh(\nu u) \))
So, in region I ($|u| < \frac{\delta}{2}$) the odd solution is $4(u) = \sinh(Ku)$
EVEN " " $4(u) = \cosh(Ku)$

In region II, $v = 0$. So

$4'' = -\varepsilon 4(u)$

if $4(u) = \sin(ku)$ = ODD
$4(u) = \cos(ku)$ = EVEN

So $4'' = -k^2 4(u) = -\varepsilon 4(u)$

Finally, in region III: $|u| > \frac{1}{2} + \frac{\delta}{2} = \frac{1}{2}(1+\delta)$

$4''(u) = (v-\varepsilon)4(u)$.

Same as region I, but row sinh(Ku) and cosh(Ku) won't work.
So now, for \( u \geq \frac{1}{2} + \frac{\sqrt{3}}{2} \) only \( e^{-Ku} \) works, and for \( u - (\frac{1}{2} + \frac{\sqrt{3}}{2}) \) only \( e^{+Ku} \) works.

* Explain why \( \sinh \) is used and \( \cosh \) won't work. [HINT: what do they both do if \( u \rightarrow \pm \infty \)]

* Explain how to make \( e^{\pm Ku} \) odd or even. Why doesn't this trick work for region 1?
So, for $u > 0$

$$\sinh(ku) \quad 0 < u < \frac{\delta}{2}$$

$$h(u) = A \sinh(ku) \quad \frac{\delta}{2} < u < \frac{1}{2} \delta$$

$$h(u) = A e^{ku} \quad \frac{1}{2} \delta < u$$

(A, B are needed to match the functions but will be unimportant)

Match the $f_{yy} = 0$ 2nd derivative @

$u = \frac{\delta}{2}$:

$$-A \sinh \left( \frac{k \delta}{2} \right) = \sinh \left( \frac{k \delta}{2} \right)$$

$$A k \cos \left( \frac{k \delta}{2} \right) = K \cosh \left( k \frac{\delta}{2} \right)$$

Calc the ratio:

$$\frac{1}{k} \tan \left( \frac{k \delta}{2} \right) = \frac{1}{K} \tanh \left( K \frac{\delta}{2} \right)$$
Now match $u = \frac{1}{2}(1 + \delta) \sin \left( \frac{k}{2} + \frac{1}{2} \delta \right) = B e^{-\frac{k}{2}(1 + \delta)}$

$A k \cos \left( \frac{k}{2} + \frac{1}{2} \delta \right) = -k \cdot B e^{-\frac{k}{2}(1 + \delta)}$

ratio:

$$\frac{1}{k} \tan \left( \frac{k}{2} + \frac{1}{2} \delta \right) = -\frac{1}{k}$$

$$\tan \left( \frac{k}{2} + \frac{1}{2} \delta \right) = -\frac{k}{k} = -\frac{1}{k} \sqrt{u - k^2}$$

check: $u = 150$, vary $\delta$

<table>
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<td>10.505</td>
<td>9.750</td>
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- Not the same (so good)
Add the other matching condition

\[
\frac{k}{K} = \frac{\tan \left(\frac{k\delta}{2}\right)}{\text{tanh} \left(\frac{k\delta}{2}\right)} = \frac{\tan \left(\frac{k\delta}{2}\right)}{\text{tanh} \left(\sqrt{v^2 - k^2} \frac{\delta}{2}\right)}
\]

\[
\tan \left(\frac{\pi}{2} + \frac{k\delta}{2}\right) = -\frac{\tan \left(\frac{k\delta}{2}\right)}{\text{tanh} \left(\sqrt{v^2 - k^2} \frac{\delta}{2}\right)}
\]

(note that as \(\delta \to 0\), this is the same as

\[
\tan \left(\frac{\pi}{2}\right) = -\frac{k}{K}
\]

which is expected

Check against earlier results: \(v = 150\)

<table>
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<tr>
<th>d</th>
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<th>0.05</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
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<td>5.137</td>
<td>4.884</td>
<td>4.626</td>
<td>4.369</td>
<td>3.139</td>
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</table>
These are not the same. The very weird when $S = 1 a 2$
For whatever reason, this makes no sense. So use the original idea:

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan \left( \frac{k}{2} + \frac{k\delta}{2} \right) = \frac{\tan \left( \frac{k}{2} \right) + \tan \left( \frac{k\delta}{2} \right)}{1 - \tan \left( \frac{k}{2} \right) \tan \left( \frac{k\delta}{2} \right)} = -\frac{k}{K}$$

Now use $\tan \left( \frac{k\delta}{2} \right) = \frac{k}{K \tanh \left( \frac{k\delta}{2} \right)}$

$$\tan \left( \frac{k}{2} \right) + \frac{k}{K \tanh \left( \frac{k\delta}{2} \right)}$$

$$\frac{1 - \tan \left( \frac{k}{2} \right) \cdot \frac{k}{K \tanh \left( \frac{k\delta}{2} \right)}}{1 - \tan \left( \frac{k}{2} \right) \cdot \frac{k}{K \tanh \left( \frac{k\delta}{2} \right)}} = -\frac{k}{K}$$

$$\frac{K \tan \left( \frac{k}{2} \right) + k \tanh \left( \frac{k\delta}{2} \right)}{K - k \cdot \tan \left( \frac{k}{2} \right) \tanh \left( \frac{k\delta}{2} \right)} = -\frac{k}{K - k^2}$$
\[ \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
k & 0 & 0.01 & 0.02 & 0.04 & 0.1 & 0.15 & 0.2 & 0.3 & 0.5 & 0.7 & 0.9 & 1.0 \\
\hline
\hline
\end{array} \]

\[ = \text{same as before} \]

And \( \varepsilon = k^2 \) decreases as \( S \) increases as shown in Harris (2nd) Problem 12 in chap 5.

Now, mark the even states.

@ \( \frac{S}{2} \):
\[ A \cos \left( \frac{kS}{2} \right) = \cosh \left( \frac{kS}{2} \right) \]

\[ -kA \sin \left( \frac{kS}{2} \right) = k \sinh \left( \frac{kS}{2} \right) \]

@ \( \frac{1}{2} + \frac{S}{2} \):
\[ A \cos \left( \frac{k}{2} + \frac{kS}{2} \right) = B e^{-\frac{k}{2}(1+S)} \]

\[ -Ak \sin \left( \frac{k}{2} + \frac{kS}{2} \right) = -kB e^{-\frac{k}{2}(1+S)} \]
\[ k \tan \left( \frac{k}{2} + \frac{k \delta}{2} \right) = K \]

\[ \tan \left( \frac{k}{2} + \frac{k \delta}{2} \right) = \frac{K}{k} = \frac{\sqrt{u-k^2}}{k} \]

\[
\frac{\tan \left( \frac{k}{2} \right) + \tan \left( \frac{k \delta}{2} \right)}{1 - \tan \left( \frac{k}{2} \right) \tan \left( \frac{k \delta}{2} \right)} = \frac{\sqrt{u-k^2}}{k}
\]

From other match: \[ \tan \left( \frac{k \delta}{2} \right) = -\frac{K}{k} \tan h \left( \frac{k \delta}{2} \right) \]

\[
\frac{\tan \left( \frac{k}{2} \right) - \frac{K}{k} \tanh \left( \frac{k \delta}{2} \right)}{1 + \frac{K}{k} \tan \left( \frac{k}{2} \right) \tanh \left( \frac{k \delta}{2} \right)} = \frac{\sqrt{u-k^2}}{k}
\]

\[
\frac{k \tan \left( \frac{k}{2} \right) - \sqrt{u-k^2} \tan h \left( \frac{u-k^2}{2} \right)}{k + \sqrt{u-k^2} \tan \left( \frac{k}{2} \right) \tanh \left( \frac{u-k^2}{2} \right)} = \frac{\sqrt{u-k^2}}{k}
\]
Now double check these: $V = 150$

$\text{mm}^4$ agree = I found a mistake

<table>
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<tr>
<th>d</th>
<th>0</th>
<th>0.01</th>
<th>0.03</th>
<th>0.04</th>
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<th>0.15</th>
<th>0.20</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
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</table>

$v$

obviously

<table>
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<tr>
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<th>0</th>
<th>1.0</th>
<th>2.0</th>
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<tr>
<td>$\varepsilon$</td>
<td>$4.706$</td>
<td>$4.706$</td>
<td>$4.706$</td>
<td>$4.706$</td>
</tr>
</tbody>
</table>

$\varepsilon$ increases as $d$ increases - as in prob. 12 of Harris

So the DESMOS page allows for calculation of the energy. And since we know $k$, we can calculate the amplitudes to get the functions.

$$A = \frac{\sinh \left(\frac{kS}{2}\right)}{\sin \left(\frac{kS}{2}\right)} \quad \text{for odd functions}$$

$$B = A e^{\frac{k}{2}(1+S)} \sin \left(\frac{k}{2}(1+S)\right)$$

$$A = \frac{\cosh \left(\frac{kS}{2}\right)}{\cos \left(\frac{kS}{2}\right)}$$

$$B = A e^{\frac{k}{2}(1+S)} \cos \left(\frac{k}{2}(1+S)\right)$$
But it's easier to just code it numerically and generate the Odd wavefunctions with initial conditions: \( 4(0) = 0, 4'(0) = 1 \)

and the Even wavefunctions with the initial conditions: \( 4(0) = 1, 4'(0) = 0 \)

So I did this in a file

`coupled-finite-box.ods`

in C:\Users\RoutineUse\Documents\University\Classes\Fall 20\PHY5351