Semimfinite Well  Analytical.

Tried a Modern route - no success
try again w/ series Sum see about
truncating the series (didn't work - reverted
to old method, used
DESMOS to (compare w spread sheet
- success! -)

\[-\frac{x^2}{2m} f''(x) + V(x)f(x) = E f(x)\]

Try \[V(x) = \begin{cases} 
  \infty & -\infty < x < 0 \\
  -V_0 & 0 < x < a \\
  0 & a < x
\end{cases}\]

\[f''(x) - \frac{2mV(x)}{x^2} f(x) = -\frac{2mE}{x^2} f(x)\]

Let \[u = \frac{x}{a}\]
\[\frac{d}{dx} = \frac{a}{x} \frac{d}{du} \quad \frac{1}{2} \frac{d}{du} = \frac{1}{2} a \frac{d}{du}\]

\[\left(\frac{d}{dx}\right)^2 = \left(\frac{a}{x} \frac{d}{du}\right)^2 = \frac{1}{2} a^2 \left(\frac{d}{du}\right)^2\]

\[\frac{1}{a^2} f''(u) = \frac{2mV_0}{x^2} f(u) = -\frac{2mE}{x^2} f(u)\]
\[ 4''(u) - \frac{2m^2 V(u)}{\hbar^2} 4(u) = - \frac{2m^2 E}{\hbar^2} 4(u) \]

Note: \[ \frac{\hbar^2}{2m^2} \rightarrow \text{GS energy of particle in box of size } a. \]

Natural scaling of energy even though this is not a regular box.

\[ E_1 = \frac{\hbar^2}{2m^2} \]

So \[ 4''(u) = V(u) 4(u) = - \varepsilon 4(u) \]

\[ \nu = \frac{V(u)}{E_1} < 0 \quad \varepsilon = \frac{E}{E_1} < 0 \quad \text{bound} \]

\[ > 0 \quad \text{free} \]

\[ 4''(u) = (\nu - \varepsilon) 4 = -(\nu_0 + \varepsilon) 4 \]

+ correlative if \[ \nu > \varepsilon \quad (w/4^+) \]

- if \[ \nu < \varepsilon \quad (-w/4^+) \]
Strictly, curvature is opposite sign of $y$ if $u < \varepsilon$

\[ y \]

Curvature same if $u > \varepsilon$

Meanwhile try a series solution

\[ y(u) = \sum_{n=1}^{\infty} c_n u^n \quad u < 1 \]

\[ = \sum_{n=1}^{\infty} b_n u^n \quad u > 1 \]

These will be the problem, $\delta$ at $n = 1$, since $y(0) = 0$.

? Does that mean no $b_0$?
\[ y''(u) - v(u) y'(u) = - \varepsilon y(u) \]

\[ u = \frac{d^2}{dx^2} \]

\[ v(u) = \frac{V(u)}{E_1} \]

\[ \varepsilon = \frac{K}{2\text{ma}^2} \]

- And in the core:

\[ u(u) = \begin{cases} 
-\frac{5}{6} & 0 < u < 1 \\
0 & 1 < u 
\end{cases} \]

\[ u < 1: \quad y''(u) = -(v_0 + \varepsilon) y(u) \]

- If $\varepsilon > 0$ (free) then curvature opposite of value of $y(u)$.

- If $\varepsilon < 0$ (bound) then curvature opposite = ONLY if $u > 1$.

Curvature same if $u < 1$.

(energy below bottom of well - exponential growth)
\[ u \geq 1: \quad \Psi''(u) = - \varepsilon \Psi(u) \]

If \( \varepsilon > 0 \) (free) curvature opposite \( \Psi'' \)
so \( \frac{1}{\varepsilon} \) for all u

- \( \varepsilon < 0 \) (bound) curvature same
  so \( \frac{1}{\varepsilon} \) or +

\[ \begin{align*}
  \text{BC's} & \quad \Psi(0) = 0 \\
  \Psi'(0) & \to 0 \\
  \Psi'(1) & = \Psi(1) \\
  \Psi''(1) & = \Psi'(1) \\
  4 \text{ Eqns.} & \quad 4 \text{ Unknowns}
\end{align*} \]

Should be undetermined by 1 (normalized?)

Anyway

\[ u < 1: \quad \Psi''(u) = -(u + 2) \Psi(u) \]

\[ \Psi(u) = A \sin \left( \frac{k}{u} \right) + B \cos \left( \frac{k}{u} \right) \]

\[ \Psi(0) = 0 = B \]

\[ \Psi(u) = A \sin \left( \frac{k}{u} \right) \]

\[ \Psi''(u) = k^2 \Psi(u) = -(u + 2) \Psi(u) \]

\[ u \geq 1: \quad \Psi''(u) = - \varepsilon \Psi(u) = +131 \Psi(u) \quad \text{if} \quad \varepsilon > 0 \]

\[ \Psi(0) = C e^{\frac{k}{u}} + D e^{-\frac{k}{u}} \]
\[ y''(u) = A e^{-kx} \quad y' = kA \cos kx \quad y'' = -k^2 A e^{-kx} \]

\[ y(u) = Ce^{kx} + De^{-kx} \]
\[ y'(u) = kCe^{kx} - kDe^{-kx} \]
\[ y''(u) = k^2Ce^{kx} + kDe^{-kx} = k^2 \]

\[ y''(u) = -k^2 A e^{-kx} = -k^2 A e^{-kx} \]

\[ k^2 = u_0 + \varepsilon \quad (\varepsilon = u_0 - 1) \]

\[ k = \sqrt{-u_0 + \varepsilon} = i\sqrt{u_0 + \varepsilon} \]

\[ y(0) = y(0) \quad A \sin k = Ce^k + De^{-k} \]
\[ y'(0) = y'(0) \quad A k \cos k = C e^k - Dk e^{-k} \]

\[ A \sin k + k \cos k = C(k+1)e^k \]

\[ C = A \frac{e^{kx} + k \cos k}{(k+1)e^k} \]

\[ A \sin k = Ce^k + De^{-k} \quad D = Ae^{k \sin k} = Ce^{2k} \]

\[ = Ae^{k \sin k} - A \frac{e^{kx} + k \cos k}{(k+1)e^k} \cdot e^{kx} \]
\[ y(1) = y(1) \quad A \sin k = C e^k + D e^{-k} \]
\[ y'(1) = y'(1) \quad A k \cos k = k C e^k - k D e^{-k} \]
\[ A \frac{k}{k} \cos k = C e^k - D e^{-k} \]

Add Equations:
\[ A (\sin k + \frac{k}{k} \cos k) = 2 C e^k \]

\[ C = \frac{1}{2} A (\sin k + \frac{k}{k} \cos k) e^{-k} \]

Subtract Equations:
\[ A (\sin k - \frac{k}{k} \cos k) = 2 D e^k \]

\[ D = \frac{1}{2} A (\sin k - \frac{k}{k} \cos k) e^{+ik} \]

Check by plugging in for C, solve for D
\[ D = C e^{2k} - A \frac{k}{k} \cos k e^k = \frac{1}{2} A (\sin k + \frac{k}{k} \cos k) e^k e^{+ik} - A \frac{k}{k} \cos k e^k \]

\[ = \frac{1}{2} A (\sin k - \frac{k}{k} \cos k) e^k \]
to get convergence

- $y(u)$ is NORMALIZABLE

require $C \to 0$ or (since $A=0$ untenable)

$sink + \frac{k}{K} cosk = 0$ or $tanh k = -\frac{k}{K}$

recall $k = \sqrt{v_0 - 131}$

$K = \sqrt{131}$

And if $sink = -\frac{k}{K} cosk$

$$D = \frac{1}{2} A (sink - (-sink)) e^k = A sink e^k$$

as before

$$k^2 = v_0 - 131 = k_0^2 - 131 = k_0^2 - K^2$$

$$K^2 = 131$$

or

$$K^2 = K_0^2 - k^2$$

$$tanh k = \frac{-k}{\sqrt{k_0^2 - k^2}} = \frac{-1}{\sqrt{(\frac{v_0}{k_0})^2 - 1}}$$
\[ k_0 = 3.5 \]

\[ k_1 = 4.8772 \]
\[ k_2 = -2.18 \]
\[ k_3 = 0.78 \]

- after 40 - no convergence.
  - try 1.5 ?
  - 1.5's no after 30 —

Go back to others: 
\[ k_c = \arctan\left( \frac{-1}{\sqrt{\left( \frac{k_0}{k_c} \right)^2 - 1}} \right) \]

DESMOS: plot \( \tan(x) \) vs \( \frac{-x}{\sqrt{16 - x^2}} \)

Intercept intersect in \( x = 2.475 \)
\[ x^2 = 6.1256 \]

Put in \( v_0 = 16 \) in spreadsheet,
get \( E = 9.87458 \)

And \( v_0 - k^2 = 16 - 6.1256 = 9.874 \)

\[ \text{Woo hoo} \]
Note that \( \frac{-x}{\sqrt{16 - x^2}} \)

is asymptotic to \( x = 4 \)

and \( \tan \alpha \) is asymptotic to \( \frac{\pi}{2} \), \( \frac{3\pi}{2} = 4.7 \)

So only 1 bound state —
to get another bound state, need

\( V_0 > (4.7)^2 = 22.0 \)

Try \( V_0 = 5^2 = 25 \) (should be 2 states)

Desmos

\( k = 2.596, 4.906 \) (near the top)

\( E = V_0 - k^2 = 25 - (2.596)^2, 25 - (4.906)^2 \)

\( = 18.26, 0.4316 \)

Spread \( (18.25969) (0.90935?) \)

\( V_0^2 = 8^2 = 64 \) (should be 3 states)

Desmos: 2.786, 5.521, 7.957

\( E = V_0 - k^2 = 56.238, 33.519, 0.686151 \)

56.23520973, 33.46073, 0.9634 ? Spreadsheet.
No bound states if \( k_0 < \frac{\pi}{a} = 1.57 \)

\[ a \times k_0^2 = v_0 = 2.4674 \]

Try \( v_0 = 2.5 \)

\[ \text{DESMOS } k = 1.581 \]

\[ \varepsilon = v_0 - k^2 = 4 \times 10^{-4} \quad (\text{so zero}) \]

Spreadsheet

Wave function near really goes to zero

= almost not bound! \( \ddagger \)

Try \( v_0 = 5 \)

\[ \text{DESMOS } k = 2.017 \]

\[ \varepsilon = 5 - (2.017)^2 = 0.9317 \]

Spread \( \varepsilon = 0.4305 \)

Seems to work ok, but not exact agreement - but I'm stopping

because - bored.