H's easier to use rapidity:

\[ \beta_B = -0.8c, \text{ so } \theta_{B}^{\text{Ann}} = \tanh(\beta_B) \]

\[ \theta_{B}^{\text{Ann}} = -1.0986 \]

\[ \beta_C = -0.9c \quad \theta_{C}^{\text{Ann}} = -1.4722 \]

\[ \theta_{C}^{\text{Ann}} = \theta_{C}^{\text{Ann}} - \theta_{B}^{\text{Ann}} = -0.3736 \]

(Should it still be negative and smaller?) (Ans: Yes)

And so \[ \beta_C = \tanh(-0.3736) \]

\[ = -0.3571 \]

2) \[ \theta_A^{\text{Bob}} = -\theta_{B}^{\text{Ann}} = -(-1.0986) \]

\[ = +1.0986 \]
relative speed of Ann \& Carl (to Bob):

$$\Delta \theta = \theta_c - \theta_A$$

$$= -0.3736 - 1.0986$$

$$= -1.4722$$

$$\Delta \beta = \tanh(-1.4722)$$

$$= -0.90000$$

make sense?

(This is SO much easier w/ rapidity)
Thus a better way to do this, but we can start with their way. The velocity 4-vector for this ray can be written \( u^\mu = \begin{pmatrix} c \\ c \cos \theta \\ c \sin \theta \\ 0 \end{pmatrix} \)

**Aside:** Calculate \( \gamma v u^\mu u^\nu \). Does it make sense? [Ans: \( \gamma v u^\mu u^\nu = 0 \)]

Now, “boost” this to another frame.

\[
u^\mu' = N^\mu_\nu u^\nu = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ -v_\mu & 1 & 0 & 0 \\ -v_\nu & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} c \\ c \cos \theta \\ c \sin \theta \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix}
\gamma c (1 - \beta \cos \theta) \\
\gamma c (\cos \theta - \beta) \\
\gamma c \sin \theta \\
0
\end{pmatrix}
\]

**Aside:** Is this 4-vector invariant? [Ans: Yes – make sure you can show this]
Treat this as an object travelling along $\mathbf{r}$, with $x, y$ components of velocity. Then products:

$$\tan \theta' = \frac{uy'}{ux'} = \frac{c \sin \theta}{c \cdot x (\cos \theta - \beta)}$$

$$= \frac{\sin \theta}{\delta (\cos \theta - \beta)}$$

Light for which $\theta = 0$ travels in the $+x$ direction, and so, to enter my eye, must come from directly behind me. As $\theta$ increases, the light comes from angles away from behind me.

Example: $\frac{\sqrt{r}}{\theta}$
In $S$, light from "behind me" is light for which $0 < \theta < \frac{\pi}{2}$ or $\cos \theta > 0$.

Similarly, light from "behind me" in $S'$ is light for which $0 < \theta < \frac{\pi}{2}$ or $\tan \theta > 0$. For the most part, this is the same as in $S$, in that light from behind in $S$ is also light "behind" $S'$.

Although at a different angle (clearly $\theta' \neq \theta$, right?).

But in $S'$, light is only behind me if $\cos \theta' > \beta$ (not $\cos \theta > 0$).
Here that is, there are some light sources behind me in S that are ahead of me in S'. Specifically, rays for which $0 < \cos \theta < \beta$ (to $\beta \to 0$, there is no such ray) and as $\beta \to 1$ all rays (except $\theta = 0$ - DIRECTLY Behind) are ahead of me.

$= \text{Plot } \theta' \text{ vs } \theta \text{ for various } \beta$

to see what happens =
A better way to do this is to look at the Wikipedia article on wave vectors, recognize that the wavevector of a light wave can be written as a 4-vector:

\[ \mathbf{k}^\mu = (k^0, k^1, k^2, k^3) = \left( \frac{\omega}{c}, k_x, k_y, k_z \right) \]

In this case,

\[ \mathbf{k}^\mu = \left( \frac{\omega}{c}, k \cos \theta, k \sin \theta \right) \]

In \( S' \), \( \mathbf{k}'^\mu = \Lambda^\mu_\nu k^\nu \)

\[ \Lambda = \begin{pmatrix} \sigma & -\sigma \beta & 0 & 0 \\ -\sigma \beta & \sigma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\omega}{c} \\ k \cos \theta \\ k \sin \theta \\ 0 \end{pmatrix} \]
Use $k = \frac{\omega}{c}$

$k' = \begin{pmatrix} 
\gamma \left( \frac{\omega}{c} \right) (1 - \beta \cos \theta) \\
\gamma k \left( \cos \theta - \beta \right) \\
k \sin \theta \\
0
\end{pmatrix}$

$\tan \theta' = \frac{k_y'}{k_x'} = \frac{\sin \theta}{\gamma (\cos \theta - \beta)}$

As before, what's interesting now is I can talk about the Doppler shift as well.
Moving @ speed $\beta$, these stars appear ahead of me. The faster I go, the more stars "bunch up" ahead. Only stars DIRECTLY behind me ($\sin \theta = 0$) stay behind. But consider the Doppler shift.
\[ \omega' = \gamma \cdot \omega (1 - \beta \cos \theta) \]. First interesting point is that stars that were at \( \theta = \frac{\pi}{2} \) are still Doppler shifted:

\[ \omega' = \gamma \omega \]

This is different from classical Doppler effect, when these sources are NOT Doppler shifted. But it's not that weird, b/c these sources are no longer \( \theta = \frac{\pi}{2} \), but rather at:

\[ \tan \theta' = \frac{\sin \frac{\pi}{2}}{\gamma (1 - \beta)} = -\frac{1}{\gamma \beta} \]

\( \theta' > \frac{\pi}{2} \) (How much greater depends on \( \beta \)). There is a source that, in \( S' \) is NOT Doppler shifted:

\[ \omega' = \gamma \omega (1 - \beta \cos \theta) = \omega \]
Given \( 1 - \beta \cos \theta = \frac{1}{\delta} \) so \( \cos \theta - \beta = \frac{\delta (1 - \beta^2) - 1}{\delta \beta} \)

\[
\tan \theta' = \frac{\sin \theta}{\delta (\cos \theta - \beta)} = \frac{\delta \sin \theta}{\delta (\cos \theta - \beta)}
\]

\[
= \frac{\beta \sin \theta}{\sqrt{1 - \beta^2} - 1}
\]

with \( \sin \theta = \sqrt{1 - \cos^2 \theta} \)

**Aside:** Start with \( \delta (1 - \beta \cos \theta) = 1 \)

\[
\cos \theta = \frac{1}{\beta} (1 - \frac{1}{\delta}) = \frac{1}{\beta} (1 - \sqrt{1 - \beta^2})
\]

As \( \beta \to 0 \)

\[
\cos \theta = \frac{1}{\beta} (1 - (1 - \frac{1}{2} \beta^2)) = \frac{\beta}{2} \approx 0
\]

Expand \( \cos \theta \) near \( \theta = \frac{\pi}{2} \)

\[
\cos \theta \approx \cos \left( \frac{\pi}{2} \right) + \frac{1}{2} \left( -\sin \frac{\pi}{2} \right) (\theta - \frac{\pi}{2}) \approx \frac{\pi}{2} - \theta
\]

So \( \cos \theta \approx \frac{\beta}{2} \) given \( \theta \approx \frac{\pi}{2} - \frac{\beta}{2} \)

\[
\cos^2 \theta = \frac{1}{\beta^2} (1 - \sqrt{1 - \beta^2})^2
\]

\[
= \frac{1}{\beta^2} \left( 1 - 2 \sqrt{1 - \beta^2} + (1 - \beta^2) \right)
\]
Do an expansion of \(\cos^2 \theta\) to get \(\cos^2 \theta \approx \frac{1}{4} \beta^2\) as expected.

\[
1 - \cos^2 \theta = 1 - \frac{1}{\beta^2} (1 - 2 \sqrt{1 - \beta^2} + (1 - \beta^2))
\]

\[
= \frac{\beta^2 - 1 + 2 \sqrt{1 - \beta^2} - 1 + \beta^2}{\beta^2}
\]

\[
= \frac{2 \beta^2 - 2 + 2 \sqrt{1 - \beta^2}}{\beta^2}
\]

\[
= \frac{2}{\beta^2} (\sqrt{1 - \beta^2} - (1 - \beta^2))
\]

Do Taylor series expansion to get \(1 - \cos^2 \theta \approx 1 - \frac{1}{4} \beta^2\) as expected.

So \(\sin \theta = \frac{\sqrt{2}}{\beta} \left[ \sqrt{1 - \beta^2} - (1 - \beta^2) \right]^{1/2}\)
so now, \( \tan \theta' \) gives:

\[
\tan \theta' = \frac{\sin \theta}{\tau (\cos \theta - \beta)}
\]

\[
= \frac{\frac{\sqrt{2}}{\beta} \left[ \sqrt{1-\beta^2} - (1-\beta^2) \right]}{\sqrt{1-\beta^2} - 1}
\]

Show that this is in the 2\textsuperscript{nd} quadrant (i.e., \( \tan \theta' < 0 \)) (i.e., ahead in \( S' \))

\[
= -\sqrt{2} \cdot \frac{\left[ \sqrt{1-\beta^2} - (1-\beta^2) \right]}{1 - \sqrt{1-\beta^2}}
\]

\( \text{Taylor series} \quad \tan \theta' \approx -\frac{2}{\beta^2} - \frac{1}{4} \beta^2 \)
Recall that as $\beta \to 1$, all sources "bunch up" ahead of me. Only stars directly behind stay there. Recall $w' = \sigma w (1 - \beta \cos \Theta)$.

For sources behind me, $\Theta = 0$ so

$$w' = \sigma w (1 - \beta) = \omega \sqrt{\frac{1 - \beta}{1 + \beta}}$$

If $\beta \to 1$, $w' \to 0$ so the sources that DO stay behind me redshift so much I can't see them.

What is $\beta$ for visible light to redshift into a radio wave?
\[ \text{ANS: } \omega' = \frac{\omega}{\sqrt{1 - \beta}} = \frac{\omega}{\sqrt{1 - \frac{1}{1 + \beta}}} = \frac{\omega}{\sqrt{1 - \frac{1 - (10^9)^2}{1 + (10^9)^2}}} = \frac{\omega}{\sqrt{1 - \frac{1 - 10^{18}}{1 + 10^{18}}}} = \frac{\omega}{\sqrt{1 - 10^{-18}}} \]

So really close to 1

Meanwhile ALL the sources that have moved ahead have also blue shifted

\[ \omega' = 8\omega [1 - \beta(-1)] = \sqrt{\frac{1 + \beta}{1 - \beta}} \omega \]

How fast for infrared to shift to X-rays?

\[ \text{ANS: } \beta = \frac{1 - (\frac{\omega}{\omega_0})^2}{1 + (\frac{\omega}{\omega_0})^2} = 1 - 10^{-8} \]
This is similar to 64, but the source is emitting in $S'$ and we want to know what $S$ sees:

$$k' = \begin{pmatrix} \frac{w'}{c} \\ k' \cos \theta' \\ k' \sin \theta' \end{pmatrix}$$

$$k'' = \Lambda_{\beta} \cdot k' = \begin{pmatrix} 1 \\ \beta \beta \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{w'}{c} \\ k' \cos \theta' \\ k' \sin \theta' \end{pmatrix}$$

$$= \begin{pmatrix} \frac{w'}{c} (\frac{w'}{c} + \beta k' \cos \theta') \\ \frac{w'}{c} (\beta \frac{w'}{c} + k' \cos \theta') \\ k' \sin \theta' \end{pmatrix} \cdot \beta k' = \frac{w'}{c}$$

$$= \begin{pmatrix} \frac{w'}{c} (1 + \beta \cos \theta') \\ \frac{w'}{c} k' (\beta + \cos \theta') \\ k' \sin \theta' \end{pmatrix}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{k' \sin \theta'}{\beta k' (\cos \theta' + \beta)} = \frac{\sin \theta'}{\beta (\cos \theta' + \beta)}$$
Do as before, and (in 64) explain the author's choice of the phrase "head light effect."

Hint:
\[ \cos \theta' = -\beta \]

- These rays stay ahead, but "bunch up" towards straight ahead and blue shift.
- As \( \beta \rightarrow 1 \), all rays rotate to point ahead.