45 and 46 - Problem 45 shows that requiring that $y(x)$ be normalized is the root of energy quantization (rather than matching the WF-continuity). 46 shows that you should use symmetry whenever you can. So I'll do both (kinda) by rewriting the whole problem as symmetric rather than doing prob 45. So:

$$U(x) = \begin{cases} 
+U_0, & x < -\frac{L}{2} \\
0, & -\frac{L}{2} \leq x < \frac{L}{2} \\
+U_0, & x > \frac{L}{2} 
\end{cases}$$

In region I ($x \leq \frac{L}{2}$)

$$-\frac{h^2}{2m} \frac{d^4}{dx^4} y(x) + U_0 y(x) = E y(x)$$

$$y''(x) = \frac{2m}{h^2}(U_0 - E) y(x) = x^2 y(x)$$

So $y(x) = C e^{\alpha x} + D e^{-\alpha x}$ for $x \leq -\frac{L}{2}$

And $y''(x) = F e^{\alpha x} + G e^{-\alpha x}$ for $x > +\frac{L}{2}$
The in region II \((-\frac{1}{2} < x < \frac{1}{2})\):

\[-\frac{\hbar^2}{2m} \psi''(x) = E \psi(x) \Rightarrow \psi''(x) + \frac{2mE}{\hbar^2} \psi(x) = -k^2 \psi(x)\]

So, \(\psi(x) = A\sin(kx) + B\cos(kx)\)

**BUT** since the WF must be either even or odd (since \(\psi(x)\) is even), we know that one of \(A\) and \(B\) is zero (but not both – why not both?)

This is how one constant is eliminated in Prob 46.

So, start w/ the condition that \(A = 0\) (we choose the even WF)

Then \(\psi:\)

\[\psi(x) = C e^{\frac{\hbar k}{2}} + D e^{\frac{-\hbar k}{2}} \quad \text{at} \quad x = -\frac{1}{2}\]

\[\psi'(x) = -kB \sin \left(\frac{\hbar k}{2}\right) = \alpha \left( C e^{\frac{\hbar k}{2}} - D e^{\frac{-\hbar k}{2}} \right) \quad \text{at} \quad x = -\frac{1}{2}\]

\[\psi(x) = F e^{\frac{\hbar k}{2}} + G e^{\frac{-\hbar k}{2}} \quad \text{at} \quad x = \frac{1}{2}\]

\[-kB \sin \left(\frac{\hbar k}{2}\right) = \alpha \left( F e^{\frac{\hbar k}{2}} - G e^{\frac{-\hbar k}{2}} \right) \]

...
multiply by \( \frac{3}{2} \) (use \( \cos(-x) = \cos(x) \), \( \sin(-x) = -\sin(x) \))

\[
x B \cos\left(\frac{4\pi}{2}\right) = x C e^{-\frac{3\pi}{2}x} + x D e^{-\frac{3\pi}{2}x}
\]

Subtract \( B \left[ x \cos\left(\frac{4\pi}{2}\right) - k \sin\left(\frac{4\pi}{2}\right) \right] = 2 x D e^{-\frac{3\pi}{2}x} \)

\[
D = \frac{B}{2} \left[ 1 - \frac{k}{\alpha} \tan\left(\frac{\pi}{2}\right) \right] \cos\left(\frac{\pi}{2}\right) e^{-\frac{3\pi}{2}x}
\]

Doing the same to find \( F \):

\[
F = \frac{B}{2} \left[ 1 - \frac{k}{\alpha} \tan\left(\frac{\pi}{2}\right) \right] \cos\left(\frac{\pi}{2}\right) e^{-\frac{3\pi}{2}x}
\]

(Which is really the same result.)

By invoking symmetry at the beginning, this was built in.

If \( F = D = 0 \) are allowed to be non-zero, there are 2 undetermined constants, (only one can be set by normalization), (which won’t work since if \( F = D = 0 \), the WE can’t be normalized). And if \( F = D = 0 \), then

\[
\tan\left(\frac{\pi}{2}\right) = \frac{\alpha}{k}, \text{which sets the EVEN WF energies}
\]
Now, for the odd WP's (B=0)

4:\[ A \sin(-\frac{1}{2}) = C e^{\frac{x}{2}} + D e^{\frac{a}{2}} \quad \text{as} \quad x = -\frac{L}{2} \]

\[ kA \cos(-\frac{1}{2}) = xCe^{\frac{x}{2}} - xDe^{\frac{a}{2}} \]

\[-A + 8 \sin(\frac{1}{2}) = xCe^{-\frac{x}{2}} + xDe^{-\frac{a}{2}} \]

Subtract:

\[-A(k \cos \frac{1}{2} + x \sin \frac{1}{2}) = 2xDe^{\frac{a}{2}} \]

\[ D = -\frac{A}{2} \left[ \frac{k}{x} + \tan \frac{1}{2} \right] \cos \frac{1}{2} e^{\frac{x}{2}} \]

As before, the WF is normalized if \( F = D = 0 \) or

\[ \tan \frac{1}{2} = -\frac{k}{x} \]

Are these energies the same as his?

\( \Box \) Solve for C, G - make sense for even is odd?
For reasons I'll go into at the end, the WF is continuous @ x=0, but the slope isn't. Using the WF we were already used to:

\[ y''(x) = -\frac{2mE}{\hbar^2} y(x) \]

And knowing that we want a bound state (\( E < 0 \)), the solution is of the form:

\[ y(x) = Ae^{-\alpha x} + Be^{\alpha x} \]

Where

\[ y''(x) = \alpha^2 y(x) = -\frac{2mE}{\hbar^2} \]

or \( -E = \frac{(\alpha \hbar)^2}{2m} \)

For the WF to be normalizable, we require \( B = 0 \) if \( x > 0 \) and \( A = 0 \) for \( x < 0 \)

For \( y \) to be continuous @ \( x = 0 \), we require \( A = B \).

So the solution is:

\[ y(x) = \begin{cases} 
A e^{\alpha x} & x < 0 \\
A e^{-\alpha x} & x > 0 
\end{cases} \]

And for no real reason, if I normalize this
\[
\int_{-\infty}^{\infty} 4(x)^2 \, dx = 2 \int_{0}^{\infty} 4(x)^2 \, dx = 2 A^2 \int_{0}^{\infty} e^{-2\alpha x} \, dx - \frac{2\alpha}{2\alpha}
\]
\[
= -\frac{A^2}{\alpha} \left( -\frac{1}{2\alpha} \right) = \frac{A^2}{\alpha} = 1 \quad \text{so} \quad \alpha = \sqrt{\alpha}
\]

Then \( f(x) = \begin{cases} 
\sqrt{\alpha} e^{-\alpha x} & x < 0 \\
\sqrt{\alpha} e^{\alpha x} & x > 0 
\end{cases} \)

\( \square \) Is this the same as his result?

b) \( f(x) \) decays exponentially away from \( x = 0 \)

\[ - \text{Aside} - \]
This potential is usually referred to as a DIRAC delta function.
It's defined simply as $\delta(x-x_0) = 0$ if $x \neq x_0$, and

$$\int_a^b \delta(x-x_0) \, dx = 0$$

if $a < x < b$, and

$$\int_a^b \delta(x-x_0) \, dx = 1$$

if $a < x < b$.

Since this function is infinitesimally narrow and has area $= 1$, it is apparently $\propto$ in height. With this, the Sch equation becomes

$$-\frac{h^2}{2m} 4''(x) - u(x) \delta(x) 4(x) = E \, 4(x)$$

For $x \neq 0$, it's what we had before and so the solution is as before (for $E < 0$, anyway).
Note that the potential is: \( U(x) = -U_0 \delta(x) \)

Where did the \( \mp \) come from? [HINT: if \( \int \delta(x) \, dx = 1 \) what are the units of \( \delta(x) \). So want are units of \( \delta \)?] \( U \) is in an energy in joules (meV)

Since \( \delta(x) = \delta(-x) \), we know our solution is either even or odd. I can integrate the Schrod equation once:

\[
y''(x) = \frac{-2mU_0}{\hbar^2} \delta(x)y(x) - \frac{2mE}{\hbar^2} y(x)
\]

\[
d(y(x)) = \frac{-2mU_0}{\hbar^2} \delta(x)y(x) \, dx - \frac{2mE}{\hbar^2} y(x) \, dx
\]

\[
\int_{-\infty}^{\infty} d(y(x)) = \frac{-2mU_0}{\hbar^2} \int_{-\infty}^{\infty} \delta(x)y(x) \, dx - \frac{2mE}{\hbar^2} \int_{-\infty}^{\infty} y(x) \, dx
\]

Where \( \epsilon \) is a small number that I well let approach zero. So I have

\[
y'(3) - y'(-3) = \frac{-2mU_0}{\hbar^2} y(0) = 0
\]
That then,

\[ 4'(0^+) - 4'(0^-) = -\frac{2mU^2}{\hbar^2} 4(0) \]

From the wavefunction:

\[ -x^2 - \alpha = \frac{2mU^2}{\hbar^2} \alpha^{\frac{3}{2}} \]

\[ \alpha = \frac{mU^2}{\hbar^2} \] And now

we know the bound energy from the potential \((U^2)\), the unbound states \((E \geq 0)\) are:

\[ 4''(x) = -\frac{2mE}{\hbar^2} 4(x) = -k_E^2 \] \((k^2 > 0)\)

So,

\[ 4(x) = A\cos k_E x + B\sin k_E x \quad x > 0 \]

\[ = C\cos k_E x + D\sin k_E x \quad x < 0 \]

\[ 4'(0^+) - 4'(0^-) = -\frac{2mU^2}{\hbar^2} 4(0) \]

\[ 4(0^+) = 4(0^-) \quad A = C \]

\[ 4'(0^+) = kB \quad 4'(0^-) = kD \]
$kB - kD = -\frac{2mU_0 A}{A^2}$  or

$B = D - \frac{2mU_0 A}{A^2}$  

(units ok?)

So I have two unknown constants and no other constants. Also, any value of $k$ is allowed (so any energy is allowed). Since these functions are not normalizable (the particle is unconstrained in its "motion") it does not satisfy $(4 \frac{d^2}{dx^2}) \to 0$ as $x \to \infty$. The usual strategy is to decide that $B = iA$. Then $4(x) = A\cos kx + iA\sin kx = Ae^{ikx}$ for $x > 0$.

$D = (i + \frac{2mU_0 A}{k^2})A$ and so

$4(x) = (1 + \frac{mU_0 A}{k^2}) Ae^{ikx} + i(\frac{mU_0 A}{k^2}) A e^{-ikx}$ for $x < 0$. 