1. **[30 points]** We solved the (1D) problem of a particle under the influence of a constant force, \((\text{in flat Minkowski space})\) starting at rest and at initial position \(x(0)\), and got 
\[ x(t) = x(0) + \left( c^2 / a \right) \left[ \sqrt{1 + \left( ai / c^2 \right)^2} - 1 \right] \].
Clearly (and intuitively), if you and a light ray take off at \(t=0\) and at \(x(0)=0\), the light ray takes off and you never catch up. But if you have a sufficiently big head start, the light ray gets closer and closer, but never passes you.

A) Show that there is a time \(t_0\) so that if you take off at \(t=0\) and at \(x(0)=0\), but the light ray is delayed \(t_0\), again, the light ray gets closer and closer but never passes. What is that time, \(t_0\)?

B) If your acceleration is equal to that provided by the Earth's gravitational field near its surface \((10 \text{m/s}/\text{s})\), how long must a light signal be delayed so that it can't catch you?

2. **[30 points]** We converted the flat space metric in Cartesian coordinates to spherical coordinates using the transforms: 
\[ z = r \cos \theta, \quad y = r \sin \theta \sin \phi, \quad x = r \sin \theta \cos \phi. \]
The conversion involved finding expressions for \(dz, dy, \) and \(dx\). I did \(dz\) in class. I also wrote out the Minkowski distance element as 
\[ ds^2 = -(c dt)^2 + (dr)^2 + r^2 (d \theta)^2 + r^2 \sin^2 \theta (d \phi)^2 \]

A) Find an expression for \(dy\) in terms of \(r, \theta, \) and \(\phi\).

B) Use \(ds^2\) (above) to write out the Minkowski (flat space) metric in spherical coordinates. [HINT: \(ds^2 = \eta_{\mu \nu} dx^\mu dx^\nu\)]

3. **[20 points]** In order to show that the science fiction idea of “warp drive” (travelling a distance \(D\) in space in a time shorter than \(D/c\)) was viable, Miguel Alcubierre postulated a distance element given by: 
\[ ds^2 = -(c dt)^2 + \left[ dx - V_s dt \right]^2 + dy^2 + dz^2 \]
(and showed that it was consistent with general relativity but without showing how such a metric could be realized).

A) Write out all 16 elements of the metric \(g_{\mu \nu}\). [HINT: Expand the squared binomial]

B) Explain why this is a curved space metric. [HINT: A space is curved if it is not flat.]

4. **[20 points]** We showed that the energy of a particle is given by \(E^2 = (mc^2)^2 + (pc)^2\) and so, in its rest frame \((p=0)\), a particle has a nonzero energy, \(E_0 = mc^2\). It's found that the mass of a composite particle like the deuteron is LESS than the total of its constituents. The interpretation of this is that the difference in MASS appears as BINDING ENERGY through \(E_0 = mc^2\). Find the binding energy of a deuteron in MeV. Show your work.
(1A) We want to show \( \frac{c^2}{a} \left[ \sqrt{1 + \left( \frac{at}{c} \right)^2} - 1 \right] > c(t-t_0) \) for all \( t > 0 \).

**Algebra:** 
\[
\sqrt{1 + \left( \frac{at}{c} \right)^2} > \frac{a}{c}(t-t_0) + 1 = \frac{at}{c} - \frac{at_0}{c} + 1
\]

\[
1 + \left( \frac{at}{c} \right)^2 > \left( \frac{at}{c} \right)^2 + \frac{a^2t^2}{c^2} + 1 + 2 \frac{at}{c} - 2 \frac{at_0}{c} - 2 \frac{at_0}{c} \frac{at_0}{c}
\]

\[
0 > 2 \frac{at}{c} \left( 1 - \frac{at_0}{c} \right) + \frac{at_0}{c} \left( \frac{at_0}{c} - 2 \right)
\]

The second term is constant. The 1st term \( A \) grows with time. If it's positive, there is a time long enough to make the inequality fail. **BUT** only if \( 1 - \frac{at_0}{c} > 0 \). Otherwise both terms are negative and the inequality holds for all time.
(A) So any $t \geq \frac{c}{a}$ works.

This $(\frac{c}{a})$ is also the only way to combine the relevant parameters ($c = a$) and get a time (in the same way that $\frac{c^2}{a^2}$ is the only way to get a distance).

**Alternately:** since I need

$$\frac{c^2}{a^2} \left[ \sqrt{1 + \left( \frac{at}{c^2} \right)^2} - 1 \right] > c(t - t_0)$$

for all time; in particular, when $t \gg \frac{c}{a}$:

$$\frac{c^2}{a^2} \left[ \sqrt{1 + \left( \frac{at}{c^2} \right)^2} - 1 \right] = \frac{c^2}{a} \left[ \left( \frac{at}{c} \right) - 1 \right] > ct - c t_0$$

so $-\frac{c^2}{a} > -c t_0$ and $t_0 > \frac{c}{a}$
1A) Finally recall that I need a headstart of \( x(0) = \frac{c^2}{a} \). This is the same as delaying the light by a time

\[
t_0 = \frac{x(0)}{c} = \frac{c}{a}
\]

3) \( 1 g = 10 \frac{\text{m/s}}{s} = 1 \frac{\text{ly/yr}}{\text{year}} = \frac{c}{\text{year}} \)

so \( t_0 = \frac{c}{a} = \frac{c}{c/\text{year}} = 1 \text{ year} \)
2) \[ dy = d(r \sin \theta \sin \phi) \]
\[ = (dr)(\sin \theta \sin \phi) \]
\[ + r(\cos \theta \, d\theta)(\sin \phi) \]
\[ + r \sin \theta (\cos \phi \, d\phi) \]

3) \[ g_{\mu \nu} = \begin{pmatrix} -1 + \left(\frac{\mu}{c}\right)^2 & -\frac{\mu}{c} & 0 \\ -\frac{\mu}{c} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \text{Multiply it out to confirm} \]
3b) The flat space metric is:

\[ g_{\mu\nu} = \text{diag} (-1, 1, 1, 1) \]  

Since \( g_{\mu\nu} \) has off-diagonal elements in some locations, the space must be curved.

4) \( \Delta E = M_0 c^2 - (m_0 + m_p) c^2 \)

\[ = (3.3436 \times 10^{-27} \text{kg})(299792458 \text{m/s})^2 \left( \frac{1 \text{eV}}{1.602 \times 10^{-19} \text{J}} \right) \left( \frac{1 \text{MeV}}{1 \times 10^6} \right) \]

\[ - (939.56 + 938.27) \text{MeV}/c^2 \]

\[ = (1875.83 - 1877.83) \text{MeV} \]

\[ = -2.00 \text{MeV} \]

(Wikipedia says 2.22 MeV)