Exam II  PHYS 331 Math Methods  Fall 2019  D Norwood

Some stuff to think about: Read all the questions first and do the ones you find easier first. Feel free to ask if something is unclear or you feel you need other information - I won't tell you how to work the problem, but I'm happy to clarify. TELL ME WHO YOU ARE. Above all, relax and say your mantra. The time has come, the walrus said....

1)  (15 points) In class, we discussed the condition $z^2 = z$, where $z$ is a complex number. It's easy enough to show that if $z^2 = z$ then $z^n = z$ . To see if it's interesting, find all complex numbers for which $z^2 = z$ . [ANS: The obvious one is $z = (0,0) = 0+0i$ . I want the others.]

2)  (20 points) Suppose we define matrix division by $\frac{\hat{A}}{\hat{B}} = \left( \sum_k \frac{a_{ik}}{b_{jk}} \right) g$ . (Recall that the element $a_{ij}$ is the row $i$, column $j$ element of matrix $A$.) Perform this "division" for the two matrices $\hat{A} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ . [ANS: I got $\begin{bmatrix} 9 & 6 \\ 4 & 6 \end{bmatrix}$ . ASIDE: Nobody does this, so presumably it's not useful.]

3)  (20 points) Prove by induction that $\int_0^\infty x^n e^{-x} dx = n!$ where $n$ is a non-negative integer.

4)  (15 points) Do the Pauli spin matrices (shown in Ch. 3, Sect. 6, Prob 6 on p. 122) form a group under matrix multiplication? You can assume that each of the things you are asked to show in Prob 6 are true. (You don't have to show them here). (HINT: I found it useful to construct the multiplication table.)

5)  (30 points) Start with the functions $\{1, x\}$ and product defined as:

$$\langle f, g \rangle = \int_0^1 \sqrt{x} f(x) g(x) dx$$

Use the Gram-Schmidt procedure to construct the first two orthonormal polynomials (I couldn't find these polynomials anywhere, so presumably they're not useful). [HINT: I found it useful to solve $\int_0^1 \sqrt{x} x^n dx$ first.]
\[ z^2 = (x + iy)^2 = x^2 - y^2 + i 2xy = x + iy = 2 \]

\[ x, y \text{ are Real} \]

So, \( 2xy = y \implies y(2x - 1) = 0 \)

**Case 1:** \( y = 0 \) \implies \( x^2 - y^2 = x \) \implies \( x^2 - x = x(x - 1) = 0 \)

\( \implies z = (1,0) \) \& \( (0,0) \)

**Case 2:** \( x = \frac{1}{2} \) \implies \( \frac{1}{4} - y^2 = \frac{1}{2} \) \implies \( y^2 = -\frac{1}{4} \)

No Real number for \( y \)

So, \( (1,0) \) and \( (0,0) \)
(2) call \( \hat{C} = \hat{A}/\hat{B} \) then \( C_{ij} = \sum_{k}^{2} a_{ik} \frac{k}{b_{jk}} \)

So, for example, \( C_{11} = \sum_{k}^{2} a_{1k} \frac{k}{b_{1k}} = \frac{a_{11}}{b_{11}} + \frac{a_{12}}{b_{12}} \)

So \( C_{11} = \frac{1}{4} + \frac{2}{1} = \frac{9}{4} \)

\( C_{12} = \frac{a_{11}}{b_{21}} + \frac{a_{12}}{b_{22}} = \frac{1}{1} + \frac{2}{4} = \frac{6}{4} \)

\( C_{21} = \frac{1}{4} + \frac{2}{1} = \frac{9}{4} \)

\( C_{22} = \frac{1}{1} + \frac{2}{4} = \frac{6}{4} \)

So \( \hat{C} = \begin{bmatrix} \frac{9}{4} & \frac{9}{4} \\ \frac{9}{4} & \frac{9}{4} \end{bmatrix} = 4 \begin{bmatrix} 9 & 6 \\ 9 & 6 \end{bmatrix} \)
(3) \( \ln n = 0 \) (the first non-negative integer)
\[
\int_0^\infty e^x dx = -e^x \bigg|_0^\infty = 0 - (-1) = 1 = 0!
\]
\[
\int_0^\infty x^n e^x dx = n!
\]
Thus,
\[
\int_0^\infty x^{n+1} e^x dx = \left(x^{n+1}\right)e^x \bigg|_0^\infty + \int_0^\infty (n+1)x^n e^x dx
\]
\[
= (n+1)\int_0^\infty x^n e^x dx = (n+1)n! = (n+1)!
\]
QED

= Technically, you have to use L'Hopital's rule for the upper limit. But a rule of thumb is that exponentials always beat polynomials (a.e. logarithms always lose).
4. \( A = (0, 1) \quad B = (0, i) \quad C = (1, 0) \)

To show it is a group, every property on p172 must hold. But to disprove it, only one property must fail. I'll form the multiplication table, but you can refute that they form a group without it.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>iC</td>
<td>iB</td>
</tr>
<tr>
<td>B</td>
<td>-iC</td>
<td>1</td>
<td>iA</td>
</tr>
<tr>
<td>C</td>
<td>-iB</td>
<td>-iA</td>
<td>1</td>
</tr>
</tbody>
</table>

\[(AB) = (0, i) \cdot (0, -i) = (0, -1) = iC\]
\[AC = (0, i) \cdot (0, 1) = (0, -i) = iB\]
\[BC = (0, -i) \cdot (0, -i) = (0, i) = iA\]

Note that I used Probs. 6.6 to finish this table.

Now, using this for each condition:

1) Closure - fail.

2) Associative - yes; inherited from matrix mult.
3) Unit element: For a 2×2 matrix mult, unit element is \((1 \ 0)\), which is not present in the set.

4) Inverse - each element is its own inverse, so pass.

\[ = \emptyset \text{ if you add } (1\ 0) = 1 \text{ and } \begin{align*} \text{iA, iB, } & \text{0 iC to the set,} \\ \text{is it a group now?} \end{align*} \]
Using the notation in Example 6, p. 182

\[ f_0, f_1, f_2 = 1, x, x^2 \]

and \( \langle f_i, g \rangle = \int_0^1 f(x)g(x)\,dx \)

They will all be of the form:

\[
\int_0^1 x^n \,dx = \frac{x^{n+\frac{3}{2}}}{n + \frac{3}{2}} - \frac{1}{6} = \frac{2}{2n+3}
\]

so \( p_0 = f_0 = 1 \)

\[
\|p_0\|^2 = \frac{2}{2(0)+3} = \frac{2}{3}
\]

\[ p_0 \perp e_0 = \sqrt{\frac{3}{2}} (= p_0/\|p_0\|) \]

\[
p_1 = f_1 - e_0 \langle e_0, f_1 \rangle
\]

\[
= x - \sqrt{\frac{3}{2}} \int_0^1 \sqrt{x} \cdot \sqrt{\frac{3}{2}} \,dx = x - \frac{3}{2} \int_0^1 x^{\frac{3}{2}} \,dx
\]

\[
= x - \frac{3}{2} \cdot \frac{2}{5} = x - \frac{3}{5}
\]

\[
\|p_1\|^2 = \int_0^1 (x - \frac{3}{5})^2 \,dx = \int_0^1 \left( x^{\frac{5}{2}} - \frac{6}{5} x^{\frac{3}{2}} + \frac{9}{25} x^2 \right) \,dx
\]
\[ f'(x) = \frac{2}{7} - \frac{6}{5} \cdot \frac{2}{5} + \frac{9}{25} \cdot \frac{2}{3} \]

\[ = \frac{150 - 252 + 120}{525} = \frac{24}{525} = \frac{8}{175} \]

\[ \|p\| = \sqrt{\frac{8}{175}} \]

Finally, \[ e_1 = \frac{x - \frac{3}{5}}{\sqrt{\frac{8}{175}}} = \sqrt{\frac{7}{2}} \left( \frac{5}{2} x - \frac{3}{2} \right) \]