Using Poisson/Laplace Equation to find \( V \) of solid sphere of charge

The sphere's charge depends only on \( r \) and not \( \theta, \phi \), so the Laplacian simplifies to:

\[
\nabla^2 V(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V(r)}{\partial r} \right)
\]

Outside the sphere, \( \rho = 0 \). So the solution is \( V(r) = C_1 - \frac{C_2}{r} \) (prob 3.3)

Inside the sphere, \( \rho = \text{const} \neq 0 \).

So the D.E. is:

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dV}{dr} \right) = -\frac{\rho}{\varepsilon_0}
\]

\[
d\left( r^2 V' \right) = -\frac{\rho}{\varepsilon_0} r^2 dr
\]

\[
r^2 V' = -\frac{\rho}{3\varepsilon_0} r^3 + C_1'
\]

\[
\frac{dV}{dr} = -\frac{\rho}{3\varepsilon_0} r + \frac{C_1'}{r^2}
\]
\[ dV = -\frac{P}{4\pi\varepsilon_0} r dr + C_1' \frac{1}{r^2} dr \]

\[ V(r) = -\frac{P}{6\varepsilon_0} r^2 + C_1' \frac{r^{-1}}{1} + C_2' \]

\[ = -\frac{P}{6\varepsilon_0} r^2 + \frac{C_1'}{r} + C_2' \]

I have 4 unknown constants to find. One can be determined by gauge invariance as usual, I'll use that freedom to say:

\[ V(r) \to 0 \quad \text{as} \quad r \to \infty \quad \text{(outside)} \]

Then \( C_1 = 0 \).

Also, note that inside, \( V \to \infty \) as \( r \to 0 \). This is the form for a point charge at the origin. But there is no point charge at the origin. So to be physically consistent with the charge:

\[ C_1' = 0 \]
Another can be determined since the voltage must be continuous everywhere. In particular, \( V(R) = V(R) \)

\[
-\frac{PR^2}{\epsilon_0} + C_2' = -\frac{C_2}{R}
\]

So

\[
C_2' = -\frac{C_2}{R} + \frac{PR^2}{\epsilon_0}
\]

So

\[
V(r) = \begin{cases} 
-\frac{C_2}{r} & n > R \\
-\frac{C_2}{R} + \frac{P}{\epsilon_0} \left( R^2 - r^2 \right) & n < R 
\end{cases}
\]

It's not obvious to me at the moment how to get \( C_2 \) without knowing that a spherically symmetric charge looks like a point charge at the origin. But then:

\[
V(r) = \frac{Q}{4\pi\epsilon_0 r} \quad \Rightarrow \quad C_2 = -\frac{Q}{4\pi\epsilon_0}
\]

\[
= -\frac{Q}{4\pi\epsilon_0} \frac{4\pi R^3}{3} = -\frac{PR^3}{3\epsilon_0}
\]
Then

\[ V(r) = \begin{cases} \frac{eR^3}{3\varepsilon_0 r} & r > R \\ \frac{2}{6\varepsilon_0} (3R^2 - r^2) & r < R \end{cases} \]

\[ \textcircled{1} \text{ Check that } V(r) \text{ is continuous} \]

\[ \textcircled{2} \text{ Check that } \vec{E} = -\nabla V \]

\[ \textcircled{3} \text{ Go back and use gauge freedom to set } V(0) = 0 \]

Again check that \( V(r) \) is continuous and that \( \vec{E} = -\nabla V \)
Dr. Blanchard figured it out.

Imposing continuity of the slope of \( V(r) \) since there are no singularities in the charge density:

\[
\frac{dV}{dr} = \text{const} \frac{1}{r}
\]

\[
+ \frac{C_2}{R^2} = -\frac{P}{6\varepsilon_0} \cdot 2R \quad C_2 = -\frac{PR^3}{3\varepsilon_0}
\]

\[
C_2 = -\frac{Q}{4\pi R^2} \cdot \frac{R^3}{8\varepsilon_0} = -\frac{Q}{4\pi \varepsilon_0}
\]

Use the same method to find the potential for a spherical shell.