1) (35 points) Imagine a stick of charge, of length $L$ and centered at the origin, oriented along the $x$-axis, with a constant (magnitude) linear charge density of:

$$\lambda(x) = \begin{cases} 
+\lambda & \text{for } 0 \leq x \leq L/2 \\
-\lambda & \text{for } -L/2 \leq x \leq 0 
\end{cases}$$

A) [IN CLASS] Set up the integral to be performed to find the electric field anywhere along the $y$-axis. The integrals should only involve $x'$ and $y$.

[TAKE HOME] Perform the integral.

B) [TAKE HOME] Find the form of the field for $y>>L$. Compare the result to the field of a dipole.

2) (40 points) For the same charge distribution as Prob. 1):

A) [IN CLASS] Set up the integral to find the voltage anywhere along the $y$-axis. The integral should only involve $x'$ and $y$.

[TAKE HOME] Perform the integral.

B) [TAKE HOME] Find the form of the potential for $y>>L$. Compare the result to the potential of a dipole.

C) [IN CLASS] Explain why you can’t use the result of Prob. 2) part A) to check the answer to Prob. 1) part A).

3) (25 points) We saw that for Laplace’s equation in Cartesian coordinates, exponential functions and trig functions (which are really other exponential functions) were the solutions used (along with the boundary conditions) to find the potential for a system with Cartesian symmetry. Consider the other symmetries:

A) [IN CLASS] Find the general solution to Laplace’s equation in spherical coordinates if the voltage depends only on the coordinate $r$.

B) [IN CLASS] Find the general solution to Laplace’s equation in polar coordinates if the voltage depends only on the coordinate $s$.

C) [TAKE HOME] Find the general solution if $V$ depends only on $\theta$. 


\[ \mathbf{r} = (0, y) = \hat{x} \cdot x + y \hat{y} \]
\[ \mathbf{r}' = (x', 0) = x' \hat{x} + y' \hat{y} \]
\[ \mathbf{n} = \mathbf{r} - \mathbf{r}' = (-x', y) \]
\[ = -x' \hat{x} + y \hat{y} \]

\[ E = \int \frac{d\mathbf{q}}{4\pi \varepsilon_0 \mathbf{r}^2} = \int \frac{\lambda(x')dx'}{4\pi \varepsilon_0 (x'^2 + y^2)^{3/2}} \cdot \frac{-x' \hat{x} + y \hat{y}}{\sqrt{x'^2 + y^2}} \]

\[ = \frac{\lambda}{4\pi \varepsilon_0} \left[ \int_{-y/2}^{y/2} \left\{ \int_{0}^{y/2} \frac{-x' dx'}{(x'^2 + y^2)^{3/2}} - \int_{0}^{y/2} \frac{-x' dx'}{(x'^2 + y^2)^{3/2}} \right\} \right] \]

\[ + \int_{-y/2}^{y/2} \left\{ \int_{0}^{y/2} \frac{dx'}{(x'^2 + y^2)^{3/2}} - \int_{0}^{y/2} \frac{dx'}{(x'^2 + y^2)^{3/2}} \right\} \]

End of In Class
1(Cont) Use symmetry to argue which of these integrals you actually have to do.

For the $x$ integrals, sub $u = x'^2 + y^2$

$$du = 2x'dx'$$

For the $y$ integrals, sub $x' = y \tan \phi$

$$dx' = y \sec^2 \phi \, d\phi$$

$$\frac{2\pi}{A} = \frac{\lambda}{4\pi \eta_0} \left[ \frac{y}{4} \int \left\{ \frac{u^{3/2}}{u} \right\}_{(L/2)^2}^{(L/2)^2 + y^2} du + \frac{1}{2} \int \frac{u^{3/2}}{u} \, du \right]$$

$$+ \frac{1}{y} \left\{ \int_{\tan^{-1}(L/2u)}^{\tan^{-1}(L/2u)} \cos \phi \, d\phi - \frac{1}{y^2} \int \cos \phi \, d\phi \right\}$$
\[ \begin{aligned}
&= \frac{\lambda}{4\pi\varepsilon_0} \left[ -x \left\{ \begin{array}{c}
-\frac{1}{2} \\
\frac{L}{\sqrt{L^2 + (2y)^2}}
\end{array} \right\} \right] + \left\{ \begin{array}{c}
\frac{1}{4} \\
\frac{L}{\sqrt{L^2 + (2y)^2}}
\end{array} \right\} \right] \\
&= -x \cdot \frac{2\lambda}{4\pi\varepsilon_0 y} \left( 1 - \left[ 1 + \left( \frac{L}{2y} \right)^2 \right]^{1/2} \right)
\end{aligned} \]
B) $(1 + x)^n \approx 1 + nx \quad \text{if} \quad x \ll 1$

So,

$$
\frac{d}{dx} \approx -x \frac{2x}{4\pi\varepsilon_0 \lambda} \left[ 1 - \left(1 - \frac{1}{2} \left( \frac{L}{2y} \right)^2 \right) \right]
$$

$$
= -x \frac{2\lambda}{4\pi\varepsilon_0 \lambda} \left[ 1 + \frac{1}{2} \frac{L^2}{4y^2} \right]
$$

$$
= -x \frac{1}{4\pi\varepsilon_0 \lambda^3} \cdot \frac{\lambda L^2}{4}
$$

A dipole of charges $q = \pm \frac{\lambda L}{2}$, separated by $d = \frac{L}{2}$, would produce exactly this field.
\[ V(y) = \frac{1}{c^{2}\eta_0} \int_{-\frac{y}{2}}^{\frac{y}{2}} \frac{dx'}{[x'^2+y^2]^{\frac{3}{2}}} \]

\[ = -\frac{\lambda}{4\pi\eta_0} \int_{-\frac{y}{2}}^{\frac{y}{2}} \frac{dy'}{y \sec \phi} + \frac{\lambda}{4\pi\eta_0} \int_{\tan^{-1}\left(-\frac{L}{2y}\right)}^{\tan^{-1}\left(\frac{L}{2y}\right)} \sec \phi d\phi \]

\[ = -\frac{\lambda}{4\pi\eta_0} \ln \left[ \sec \phi + \tan \phi \right] + \frac{\lambda}{4\pi\eta_0} \ln \left[ \sec \phi + \tan \phi \right] \]
the tan limits are straightforward:

\[ \tan[A + \tan(0)] = 0 \quad \text{and} \quad \tan[A - \tan(\pm \frac{L}{2y})] = \pm \frac{L}{2y}. \]

Show that:

\[ \sec[A + \tan(\frac{L}{2y})] = \frac{\sqrt{(2y)^2 + L^2}}{2y} \]

\[ \text{and} \quad \sec[A - \tan(\frac{L}{2y})] = \frac{\sqrt{(2y)^2 + L^2}}{-2y} \]

\[ V(y) = -\frac{\lambda}{4\pi\varepsilon_0} \ln \left[ 1 + 0 \right] + \frac{\lambda}{4\pi\varepsilon_0} \ln \left[ \frac{\sqrt{2y} - \frac{L}{2y}}{-2y} \right] \]

\[ + \frac{\lambda}{4\pi\varepsilon_0} \ln \left[ \frac{\sqrt{2y} + \frac{L}{2y}}{2y} \right] - \frac{\lambda}{4\pi\varepsilon_0} \ln \left[ 1 + 0 \right] \]

\[ = 0 \quad \text{(long way to go)} \]

Use symmetry to shorten it.

B) \quad C) \quad E = -\nabla V. \quad \text{All we know is that} \quad E_h = 0
A) B) Ch 3 prob 3

\[ \frac{\partial \Phi}{\partial r} = \frac{\partial V}{\partial \phi} = 0 \quad \Rightarrow \quad \sqrt{V} = \frac{1}{r} \sin \theta \left( \sin \theta d\phi \right) \]

so \( \partial_\theta (\sin \theta d\phi V) = 0 \) and \( \sin \theta d\phi V = C_1 \)

\[ dV = C_1 \csc \theta d\theta \]

so \( V(\theta) = C_1 \ln \left[ \csc(\theta) + \cot(\theta) \right] + C_2 \)

\[ \csc \theta + \cot \theta = \frac{1 + \cos \theta}{\sin \theta} = \frac{2 \cos^2 \left( \frac{\theta}{2} \right)}{2 \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \right)} = \cot \left( \frac{\theta}{2} \right) \]

\[ V(\theta) = C_1 \ln \left[ \cot \left( \frac{\theta}{2} \right) \right] + C_2 \]