**Rotational Inertia Examples:**

By considering the kinetic energy of an object, we found that the kinetic energy can always be written as:

\[ K = \frac{1}{2} M V_{\text{COM}}^2 + \frac{1}{2} \sum m_i u_i^2 \]

where \( M \) is the total mass; \( V_{\text{COM}} \) is the speed of the center of mass; \( m_i \) is the mass of the \( i \)th particle and \( u_i \) is the speed of the \( i \)th particle as measured from the point of view of the center of mass. This is very general (works for rigid objects like baseballs as well as non rigid objects like galaxies).

In the special case that the object is rigid (all particles remain in the same position relative to all the other particles – the object “keeps its shape”), then \( u_i = r_i \omega \) and so:

\[ \sum m_i u_i^2 = \sum m_i (r_i \omega)^2 = \left( \sum m_i r_i^2 \right) \omega^2 = I \omega^2 \]

where we defined the Rotational Inertia, \( I \), (also called moment of inertia). From the kinetic energy:

\[ K = \frac{1}{2} M V_{\text{COM}}^2 + \frac{1}{2} I \omega^2 \]

we see that the kinetic energy has two terms, one due to translational (in a line) motion, which we have discussed, and one due to rotational motion. In the same way that \( \omega \), the rotational speed is analogous to the regular speed, \( v \), the rotational inertia, \( I \), is analogous to the regular inertia, \( M \). That is, the rotational inertia, \( I \), is as measure of the resistance to a change in rotational motion just as the mass is a measure of the resistance to a change in translational motion. As we discussed, we can approximate the sum of about Avogadro’s number of particles, each of an incredibly tiny mass, as an integral (that is, a nearly infinite sum of nearly infinitesimal masses is close enough to an actually infinite sum of actually infinitesimal masses):

\[ \sum m_i r_i^2 \approx \int dm r^2 \]

Example 9.4 shows you how to set this up. To compare, we can do the same thing, a uniform rod, but calculating the moment of inertia for rotation about the center (i.e., the center of mass). That is, do the same calculation as 9.4, but with the origin at the center.

\[ I = \int_{-L/2}^{+L/2} dm \int dm r^2 = \int_{-L/2}^{+L/2} \lambda \, dx \, x^2 = \lambda \left( \frac{x^3}{3} \right)_{-L/2}^{+L/2} = \frac{2}{3} \lambda \left( \frac{L}{2} \right)^3 = \frac{\lambda L^3}{12} \]

To write this in terms of the mass, \( M \), instead of the mass density, \( \lambda \), we can calculate the mass as we did when calculating the center of mass:

\[ M = \int dm = \int_{-L/2}^{+L/2} \lambda \, dx = \lambda \left( x \right)_{-L/2}^{+L/2} = \frac{\lambda L^2}{12} \]

Substitute this for \( \lambda \) and the moment of inertia becomes:

\[ I = \frac{\lambda L^3}{12} = \frac{M L^2}{12} \]

If you compare this to the result of Example 9.4, you see this result is one fourth as large. Think physically about why this would be. (HINT: The definition of \( I \) shows that mass that is twice as far from the COM is four times as effective at resisting rotational acceleration.

Next, we’ll calculate the rotational inertia of our “baseball bat” model that we used when we calculated the center of mass:
Rotational Inertia of a Baseball bat:

The key is to model the mass distribution of the bat. Before, we modeled the bat as having a mass distribution of:

\[ \frac{dm}{dx} = Kx^2 \]

where K is a constant (QUESTION: what are the units of K?)

Recall that the origin is at one end of the bat.

We will need the mass later, so to calculate the mass, we integrate \( dm \):

\[
M = \int dm = \int_0^L Kx^2 \, dx = K \left( \frac{L^3}{3} \right)
\]

Now calculate the rotational inertia as before:

\[
I = \int dm \, r^2 = \int_0^L Kx^2 \, dx \cdot x = K \int_0^L x^4 \, dx = \frac{KL^5}{5}
\]

Substituting for \( K \) from above:

\[
I = K \frac{L^5}{5} = \frac{3M L^5}{5} = \frac{3}{5} M L^3
\]

QUESTION: Compare the rotational inertia of a rod (Ex 9.4) to the rotational inertia above and explain physically why a "baseball bat" with the same M and L as the rod has about twice the rotational inertia.